Partial differential Equations :

* A partial differential equation (henceforth abbreviated P.D.E) for a functi[on] $u(x, y, \dots)$ is a relation of the form :

$$F(x_1, x_2, \dots, x_n, u_{x_1}, \dots, u_{x_n}, u_{x_1^2}, \dots\dots) = 0 \qquad (*)$$

star

* F is a given function of the independent variables $x_1, \dots, x_n$ and of the unknown function u and of a finite number of its partial derivatives.

* u is called a solution of $(*)$, if after substitution, $(*)$ is satisfied in some region Ω in the space of these indpt variables.

* Several P.D.E's involving one or more unknown functions and their derivative[s] constitute a system of equations.

* The order of a P.D.E is the order of the highest derivatives occuring in it. example $\sum_{i=1}^{n} \frac{\partial^m u}{\partial x_i^m} + Au = f$ order m.

* A P.D.E is said to be linear if it is linear in the unknown functions and their derivatives, with coefficients depending on the indpt variable only $x_1, \dots, x$[$_n$]

* A P.D.E of order m is called quasi linear in the derivatives of order m with coefficients that depend on $x_1, x_2, \dots, x_n$ and the derivatives of u of order $<$[m]

* $$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$ is called the Laplace operator

* The Laplace equation : is a linear second order equation :

$$-\Delta u = f = 0$$

* Solutions u of this equation are called potential functions or harmonic fct[s]

* the wave equation in n dimensions for $u = u(x_1, \dots, x_n, t)$ is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

(c = constant > 0) . It represents vibrations of strings or propagation of sound

* Elastic waves are described by the linear system :

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \mu \Delta u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}(\operatorname{div} u) \qquad i=\overline{1,3}$$

where $u_i(x_1, x_2, x_3, t)$ are the components of the displacement vector u, and ρ is the density and λ, μ are the Lamé constants of the elastic material.

* The heat equation (heat conduction) is $\frac{\partial u}{\partial t} = k\,\Delta u \quad (k = c^{te} > 0)$ is satisfied by the temperature of a body conducting heat, when the density and specific heat are constant.

* Surfaces corresponding to solutions of a P.D.E are called integral surfaces

* If $a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$ is a first order quasi linear P.D.E

characteristic direction

then $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right)$ and the integral surfaces are surfaces that at each point are tangent to the characteristic direction.

* Along a characteristic curve the following relation holds.

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

* The Cauchy problem for a Quasilinear P.D.E : Let Γ be a curve in xyz space represented parametrically by

$$x = f(s)\ ,\ y = g(s)\ ,\quad u = h(s)$$

* We are asking for a sol^n $u(x,y)$ of Q.L.P.D.E such that

$$h(s) = u(f(s), g(s)) \text{ holds.}$$

* Finding a solution $u(x,y)$ for a given data $f(s), g(s), h(s)$ constitutes the Cauchy problem for the Q.L.P.D.E.

* The <u>initial value</u> problem is the special Cauchy problem in which the <u>curve</u> Γ has the form $x = s$, $\underset{\substack{\uparrow \\ \text{time}}}{y = t} = 0$, $u = h(s)$

* <u>Classification</u> of 2^{end} order P.D.E's:

$$a(x, y, u, u_x, u_y)\frac{\partial^2 u}{\partial x^2} + 2b(\quad)\frac{\partial^2 u}{\partial x \partial y} + c(x, y, u, u_x, u_y) = d(x, y, \ldots$$

This equation is called if $b^2 - ac < 0$ <u>elliptic</u> and <u>hyperbolic</u> if $-ac + b^2 > 0$ and <u>parabolic</u> if $b^2 - ac = 0$

* <u>Linear partial differential operators</u>: We denote by $x = (x_1, \ldots, x_n)$ a point in $\mathbb{R}^n$ and by D_j the partial differential operator $\frac{\partial}{\partial x_j}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote n-tuple of non negative integers. We then define

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$$

Let $|\alpha|$ <u>denote</u> the <u>sum</u> of the <u>components</u> of α: $|\alpha| = \sum_{i=1}^{n} \alpha_i$

D^α is also written $= \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$

* A <u>linear partial differential equation</u> (P.D.E) of <u>order m</u> in $\mathbb{R}^n$ is an equation of the form

$$\sum_{|\alpha| \leq m} a^\alpha D^\alpha u = f$$

where a^α and f are functions of $x \in \mathbb{R}^n$. f is called the <u>right hand side</u> (r.h.s) of the equation

* The <u>principal part</u> of the general linear partial differential operator

The Heat equation

* The heat equation is the most important example of a linear partial differential equation of parabolic type.

1) <u>Heat conduction in a finite rod</u> : Let us consider the problem of determining the temperature distribution in a cylindrical rod of length L which made of homogeneous, isotropic material and has insulated cylindrical Surface. Assuming that the initial temperature distribution and the temperature at the ends ($x=0$, and $x=L$ for $t\geq 0$) are known we are faced with the initial-value problem

Find a fct $u(x,t)$ defined and cont. on the closed strip $0\leq x\leq L$ satisfy the heat eqt (1) initial cond (2) and bndry cond (3).

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} & 0<x<L \quad t>0 \quad (1) \\ u(x,0) = \phi(x) & 0\leq x\leq L \quad (2) \\ u(0,t) = f_1(t), \quad u(L,t) = f_2(t) & t\geq 0 \quad (3) \end{cases}$$

the functions ϕ, f_1 and f_2 are assumed to be continuous and satisfy the compatibility conditions $\phi(0) = f_1(0)$, $\phi(L) = f_2(0)$

<u>Theorem</u> : (1) Let T be any number >0. Suppose that the fct $u(x,t)$ is cont. in the closed rectangle R $\underbrace{\{: 0\leq x\leq L \quad 0\leq t\leq T\}}_{(x,t)}$, and satisfies the heat eqt (1). Then u attains its minimum and maximum values on the lower base $t=0$ or on the vertical sides $x=0$, $x=L$ of R

(2) There is <u>at most</u> one solⁿ of the initial boundary value problem (1)+(2)+(3).

(3) The solⁿ of the initial-boundary value problem depends countinuously on the initial and boundary data

<u>Solution of (1)+(2)+(3)</u>, : Separat of variabls $u(x,t) = X(x)\,T(t)$.
(homogeneous)
Boundary cond $f_1 = f_2 = 0$

$$X'' + \lambda X = 0 \qquad 0<x<L$$
$$X(0) = 0, \quad X(L) = 0$$

String
membrane

$$T' + \lambda T = 0 \qquad t > 0$$

and the Solⁿ takes the form:

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2}}$$

$$C_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx$$

<u>The wave equation</u>: It is the most important example of a linear second order P.DE of hyperbolic type.

<u>one dimensional wave eqt</u>: $\left(\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \right)$. Consider a taut string of Length L

1) with both ends fixed. In order to determine the vibrations of the string we must solve the initial boundary value problem

$$(1) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 < x < L \qquad t > 0$$

$$(2) \quad u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = \psi(x) \qquad 0 \le x \le L$$

$$(3) \quad u(0,t) = 0, \quad u(L,t) = 0 \qquad t \ge 0$$

this problem has a unique solⁿ. The method of separation of variables gives

$$u(x,t) = X(x)T(t) \Rightarrow$$

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < L \\ T'' + \lambda T = 0 & t > 0 \end{cases}$$

2) <u>the vibrations of a rectangular membrane</u>: Consider a stretched membrane which is fastened to a rectangular frame of length a and width b. In order to Study its Vibrations we must solve the initial boundary value problem

$$(1) \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{matrix} 0 < x < a \\ 0 < y < b \end{matrix} \quad t >$$

$$(2) \quad \begin{cases} u(x,y,0) = \phi(x,y) \\ \frac{\partial u}{\partial t}(x,y,0) = \psi(x,y) \end{cases} \qquad 0 \le x \le a, \; 0 \le y \le$$