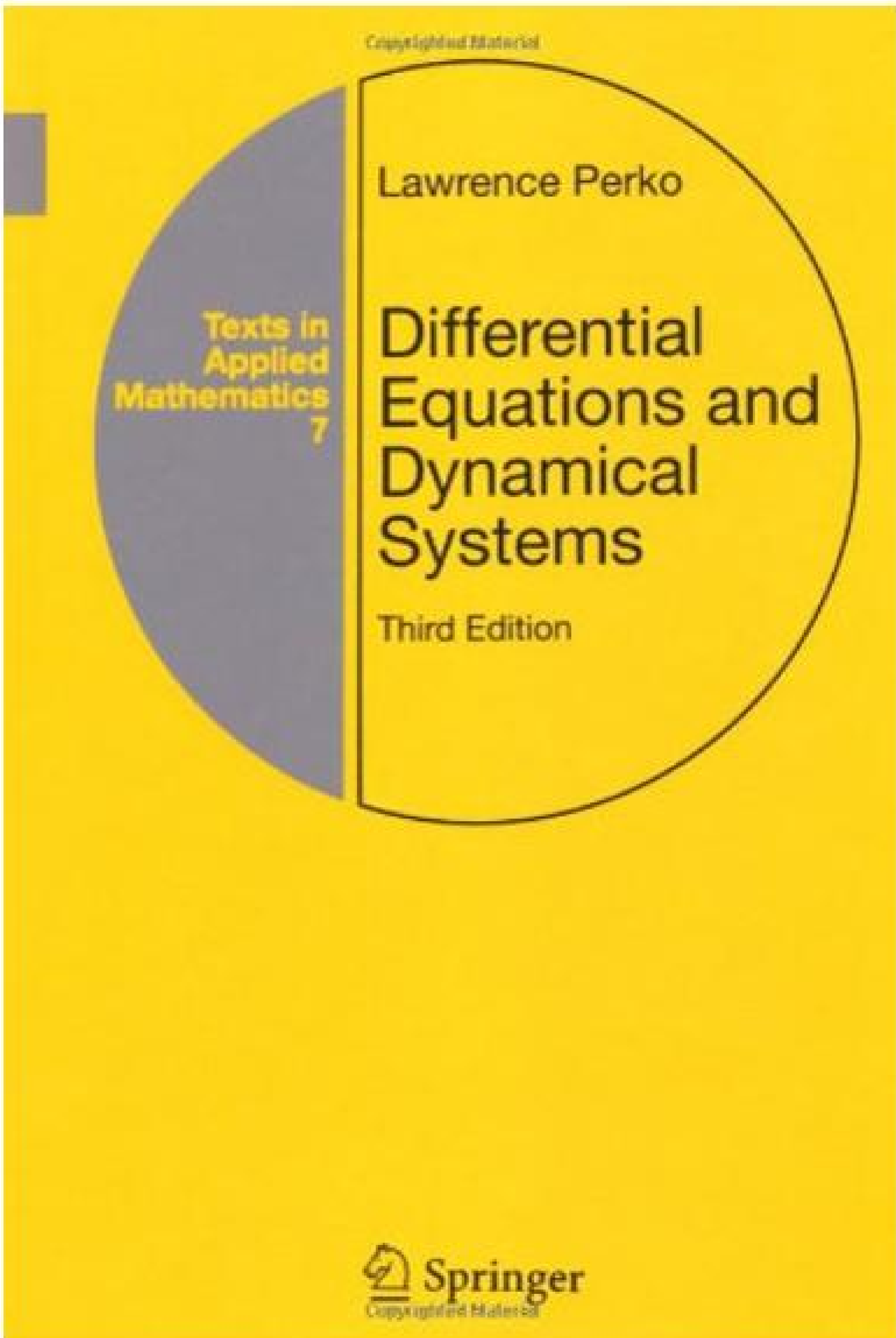
Solutions Manual

Differential Equations and Dynamical Systems

3rd Edition

Lawrence Perko



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SOLUTIONS MANUAL
for

Differential Equations and Dynamical Systems

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Springer

PREFACE

This set of problem solutions for the 3rd edition of the author's book *Differential Equations and Dynamical Systems* is intended as an aid for students working on the problem sets that appear at the end of each section in the book. Most of the details necessary to obtain the solutions, along with the solutions themselves, are given for all of the problems in the book. Those solutions not found in the main body of the solutions manual can be found in the appendix at the end of the manual.

Any additions, corrections or innovative methods of solution should be sent directly to the author, Lawrence Perko, Department of Mathematics, Northern Arizona University, Flagstaff, Arizona 86011 or to Lawrence.Perko@NAU.EDU. The author would like to take this opportunity to thank Louella Holter for her patience and precision in typing the camera-ready copy for this solutions manual and the appendix.

Also, several interesting new problems as well as a list of additions and corrections for the 3rd edition of Differential Equations and Dynamical Systems have been added at the end of the appendix in this solutions manual.

CONTENTS

1. LINEAR SYSTEMS

PROBLEM SET 1.1

Let $\mathbf{x} = (x_1, x_2, x_3)^T = (x, y, z)^T$ and $\mathbf{x}(0) = (x_0, y_0, z_0)^T$.

1. (a) $x(t) = x_0 e^t$, $y(t) = y_0 e^t$, and solution curves lie on the straight lines $y = (y_0/x_0)x$ or on the y-axis. The phase portrait is given in Problem 3 below with $a = 1$.

(b) $x(t) = x_0 e^t$, $y(t) = y_0 e^{2t}$, and solution curves, other than those on the x and y axes, lie on the parabolas $y = (y_0/x_0^2)x^2$. Cf. Problem 3 below with $a = 2$.

(c) $x(t) = x_0 e^t$, $y(t) = y_0 e^{3t}$, and solution curves lie on the curves $y = (y_0/x_0^3)x^3$.

(d) $\dot{x} = -y$, $\dot{y} = x$ can be written as $\ddot{y} = \dot{x} = -y$ or $\ddot{y} + y = 0$ which has the general solution $y(t) = c_1 \cos t + c_2 \sin t$; thus, $x(t) = \dot{y}(t) = -c_1 \sin t + c_2 \cos t$; or in terms of the initial conditions $x(t) = x_0 \cos t - y_0 \sin t$ and $y(t) = x_0 \sin t + y_0 \cos t$. It follows that for all $t \in \mathbf{R}$, $x^2(t) + y^2(t) = x_0^2 + y_0^2$ and solution curves lie on these circles. Cf. Figure 4 in Section 1.5.

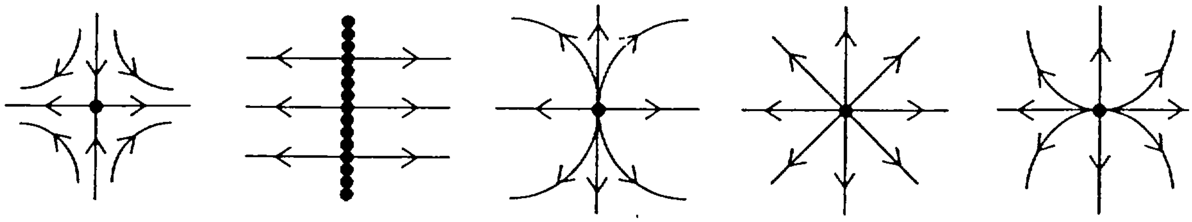
(e) $y(t) = c_2 e^{-t}$ and then solving the first-order linear differential equation $\dot{x} + x = c_2 e^{-t}$ leads to $x(t) = c_1 e^{-t} + c_2 t e^{-t}$ with $c_1 = x_0$ and $c_2 = y_0$. Cf. Figure 2 with $\lambda < 0$ in Section 1.5.

2. (a) $x(t) = x_0 e^t$, $y(t) = y_0 e^t$, $z(t) = z_0 e^t$, and $E^u = \mathbf{R}^3$.

(b) $x(t) = x_0 e^{-t}$, $y(t) = y_0 e^{-t}$, $z(t) = z_0 e^t$, $E^s = \text{Span}\,\{(1, 0, 0)^T, (0, 1, 0)^T\}$, and $E^u = \text{Span}\,\{(0, 0, 1)\}$. Cf. Figure 3 with the arrows reversed.

(c) $x(t) = x_0 \cos t - y_0 \sin t$, $y(t) = x_0 \sin t + y_0 \cos t$, $z(t) = z_0 e^{-t}$; solution curves lie on the cylinders $x^2 + y^2 = c^2$ and approach circular periodic orbits in the x,y plane as $t \to \infty$; $E^c = \text{Span}\,\{(1, 0, 0)^T, (0, 1, 0)^T\}$, $E^s = \text{Span}\,\{(0, 0, 1)^T\}$.

3. $x(t) = x_0 e^t$, $y(t) = y_0 e^{at}$.



a = −1   a = 0   a = 1/2   a = 1   a = 2

4. $x_1(t) = x_{10}e^{\lambda_1 t}$, $x_2(t) = x_{20}e^{\lambda_2 t}$, $\cdots$, $x_n(t) = x_{n0}e^{\lambda_n t}$. Thus, $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ for all $\mathbf{x}_0 \in \mathbf{R}^n$ if $\lambda_1 < 0, \cdots, \lambda_n < 0$ (and also if $R_e(\lambda_j) < 0$ for $j = 1, 2, \cdots, n$).

5. If $k > 0$, the vectors $A\mathbf{x}$ and $kA\mathbf{x}$ point in the same direction and they are related by the scale factor k. If $k < 0$, the vectors $A\mathbf{x}$ and $kA\mathbf{x}$ point in opposite directions and are related by the scale factor $|k|$.

6. (a) $\dot{\mathbf{w}}(t) = a\dot{\mathbf{u}}(t) + b\dot{\mathbf{v}}(t) = aA\mathbf{u}(t) + bA\mathbf{v}(t) = A[a\mathbf{u}(t) + b\mathbf{v}(t)] = A\mathbf{w}(t)$ for all $t \in \mathbf{R}$.

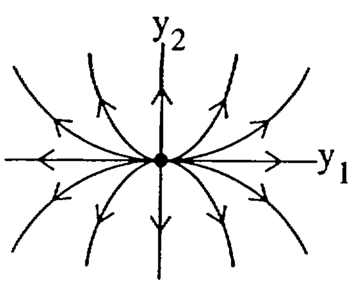
(b) $\mathbf{u}(t) = (e^t, 0)^T$, $\mathbf{v}(t) = (0, e^{-2t})^T$ and the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is given by $\mathbf{x}(t) = x_0\mathbf{u}(t) + y_0\mathbf{v}(t)$.

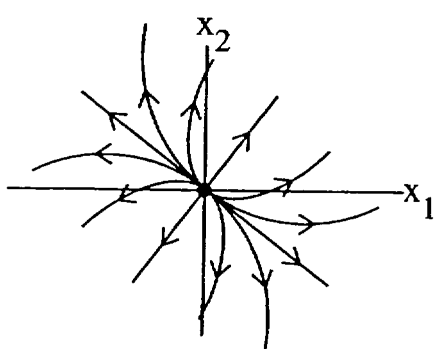
Problem Set 1.2

1. (a) $\lambda_1 = 2$, $\lambda_2 = 4$, $\mathbf{v}_1 = (1, -1)^T$, $\mathbf{v}_2 = (1, 1)^T$, $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $P^{-1} = 1/2\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and

$$B = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}\mathbf{y}_0, \qquad \mathbf{x}(t) = P\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}P^{-1}\mathbf{x}_0 = 1/2\begin{bmatrix} e^{2t}+e^{4t} & e^{4t}-e^{2t} \\ e^{4t}-e^{2t} & e^{4t}+e^{2t} \end{bmatrix}\mathbf{x}_0.$$



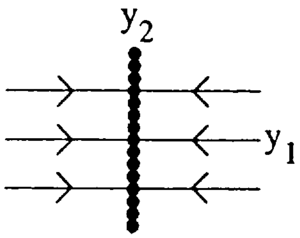


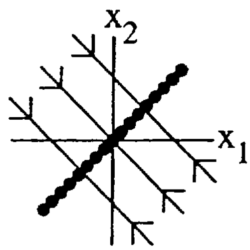
(b) $\lambda_1 = 4, \lambda_2 = -2, \mathbf{v}_1 = (1, 1)^T, \mathbf{v}_2 = (1, -1)^T$,

$$\mathbf{y}(t) = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \mathbf{y}_0, \qquad \mathbf{x}(t) = 1/2 \begin{bmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{bmatrix} \mathbf{x}_0.$$

(c) $\lambda_1 = -2, \lambda_2 = 0, \mathbf{v}_1 = (1, -1)^T, \mathbf{v}_2 = (1,1)^T$,

$$\mathbf{y}(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}_0, \qquad \mathbf{x}(t) = 1/2 \begin{bmatrix} e^{-2t} + 1 & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{bmatrix} \mathbf{x}_0.$$





2. $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1, \mathbf{v}_1 = (2, -2, 1)^T, \mathbf{v}_2 = (0, 1, 0)^T, \mathbf{v}_3 = (0, 0, 1)^T$

$$\mathbf{y}(t) = \begin{bmatrix} e^{t} & & \\ & e^{2t} & \\ & & e^{-t} \end{bmatrix} \mathbf{y}_0, \quad \mathbf{x}(t) = 1/2 \begin{bmatrix} 2e^{t} & 0 & 0 \\ 2(e^{2t} - e^{t}) & 2e^{2t} & 0 \\ e^{t} - e^{-t} & 0 & 2e^{-t} \end{bmatrix} \mathbf{x}_0,$$

$E^s = \text{Span}\,\{\mathbf{v}_3\}$, $E^u = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$.

3. $\dot{\mathbf{x}} = A\mathbf{x}$

(a) $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

(Also, see p. 121 in the appendix.)

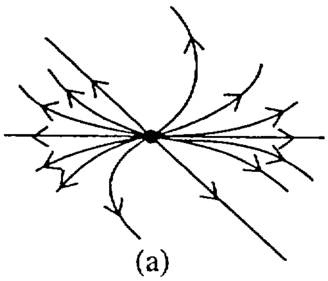
4. (a) $\mathbf{x}(t) = 1/2(3e^{4t} - e^{2t}, 3e^{4t} + e^{2t})$ (b) $\mathbf{x}(t) = 1/2(2e^{t}, 6e^{2t} - 2e^{t}, e^{t} + 5e^{-t})$.

5. $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ iff $\lambda_j < 0$ for $j = 1, 2, 3, \cdots, n$.

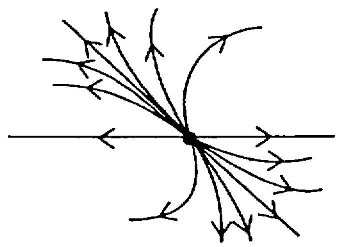
6. $$\phi(t, x_0) = P\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1} x_0 \text{ and } \lim_{y_0 \to x_0} \phi(t, y_0) = \phi(t, x_0) \text{ since } \lim_{y_0 \to x_0} y_0 = x_0$$

according to the definition of the limit.

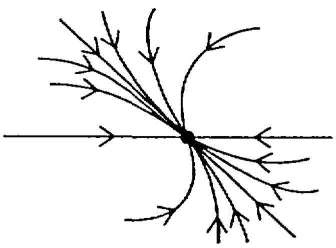
7.



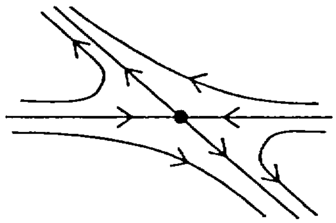
(a)



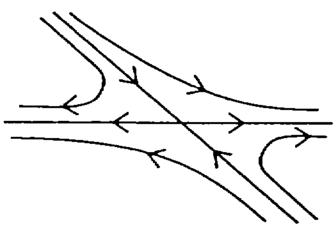
(b)



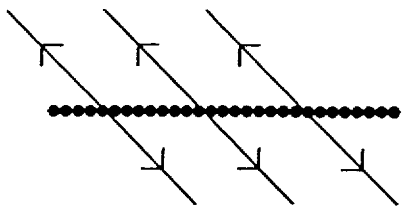
(c)



(d)



(e)



(f)

PROBLEM SET 1.3

1. (a) $\|A\| = \max\limits_{|x| \le 1} |Ax| = \max\limits_{|x| \le 1} \sqrt{4x^2 + 9y^2} \le 3|x|$; but for $\mathbf{x} = (0,1)^T$, $|A\mathbf{x}| = |-3| = 3$; thus, $\|A\| = 3$.

(b) Following the hint for (c), we can maximize $|Ax|^2 = x^2 + 4xy + 5y^2$ subject to the constraint $x^2 + y^2 = 1$ to find $x^2 = (2 \pm \sqrt{2})/4$ and $y^2 = 1 - x^2$ which leads to $\|A\| = 2.4142136\cdots$; or since $AA^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ with eigenvalues $3 \pm 2\sqrt{2}$, we have $\|A\| = \sqrt{3+2\sqrt{2}} = 1 + \sqrt{2}$.

(c) We can either maximize $|Ax|^2 = 26x^2 + 10xy + y^2$ subject to the constraint $x^2 + y^2 = 1$; or find the eigenvalues of $AA^T = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$ which are $(27 \pm \sqrt{725})/2$; in either case, $\|A\| = 5.1925824\cdots$.

2. By definition, $\|T\| = \max_{|x| \le 1} |T(x)|$. Thus, $\|T\| \ge \max_{|x|=1} |T(x)|$. But $\max_{|x|=1} |T(x)| = \sup_{x \ne 0} \frac{|T(x)|}{|x|}$ since if $|x| = a$ and we set $y = x/a$ for $x \ne 0$, then $|y| = |x|/a = 1$ and since T is linear, $\sup_{x \ne 0} \frac{|T(x)|}{|x|} = \sup_{x \ne 0} \frac{|T(x)|}{a} = \sup_{x \ne 0} \left|T(\frac{x}{a})\right| = \max_{|y|=1} |T(y)|$. Thus, $\|T\| \le \sup_{0<|x|\le 1} \frac{|T(x)|}{|x|} \le \sup_{x \ne 0} \frac{|T(x)|}{|x|} = \max_{|x|=1} |T(x)|$. It follows that $\|T\| = \max_{|x|=1} |T(x)| = \sup_{x \ne 0} |T(x)|/|x|$.

3. If T is invertible, then there exists an inverse, T^{-1}, such that $TT^{-1} = I$ and therefore $\|TT^{-1}\| = 1$. By the lemma in Section 3, $1 = \|TT^{-1}\| \le \|T\| \ \|T^{-1}\|$. This implies that $\|T\| > 0$, $\|T^{-1}\| > 0$, and $\|T^{-1}\| \ge \frac{1}{\|T\|}$.

4. Given $T \in L(\mathbf{R}^n)$ with $\|I - T\| < 1$. Let $a = \|I - T\| < 1$ and the geometric series Σa^k converges. Thus, by the Weierstrass M-Test, $\sum_{k=0}^{\infty} (I - T)^k$ converges absolutely to $S \in L(\mathbf{R}^n)$. By induction it follows that $T[I + (I - T) + \cdots + (I - T)^n] = I - (I - T)^{n+1}$. Thus, $TS = T \sum_{k=0}^{\infty} (I-T)^k = \sum_{k=0}^{\infty} T(I-T)^k = \lim_{n\to\infty} \sum_{k=0}^{n} T(I-T)^k = \lim_{n\to\infty} \left[I - (I-T)^{n+1}\right] = I$ since $\lim_{n\to\infty} \|I - T\|^{n+1} = 0$ which implies that $\lim_{n\to\infty} (I - T)^{n+1} = 0$ since $0 \le \|(I - T)^{n+1}\| \le \|(I - T)\|^{n+1}$. Therefore $S = T^{-1}$.

5. (a) $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-3} \end{bmatrix}$.

(b) The eigenvalues and eigenvectors of A are $\lambda_1 = 1, \lambda_2 = -1, \mathbf{v}_1 = (1, 0)^T, \mathbf{v}_2 = (-1, 1)^T$; thus, $e^A = P\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} e & e - e^{-1} \\ 0 & e^{-1} \end{bmatrix}$ where $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(c) $e^A = e\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ by Corollary 4.

(d) The eigenvalues and eigenvectors of A are $\lambda_1 = 2, \lambda_2 = -1, \mathbf{v}_1 = (2, 1)^T, \mathbf{v}_2 = (1, 1)^T$; thus, $e^A = P\begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix}P^{-1} = \begin{bmatrix} 2e^2 - e^{-1} & 2e^{-1} - 2e^2 \\ e^2 - e^{-1} & 2e^{-1} - e^2 \end{bmatrix}$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

(e) $e^A = e^2\begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}$ by Corollary 3.

(f) The eigenvalues and eigenvectors of A are $\lambda_1 = 1, \lambda_2 = -1, \mathbf{v}_1 = (1, 1)^T, \mathbf{v}_2 = (-1, 1)^T$; thus $e^A = P\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix} P^{-1} = \begin{bmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{bmatrix}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Note that $A^2 = I$ and from Definition 2 it therefore follows that $e^A = I(1 + 1/2! + 1/4! + \cdots) + A(1 + 1/3! + 1/5! + \cdots) = I\cosh(1) + A\sinh(1)$. This remark also applies to part (b).

6. (a) The eigenvalues are e^2, e^{-3}; e, e^{-1}; e, e; e^2, e^{-1}; $e^{2\pm i} = e^2[\cos(1) \pm i\sin(1)]$; e, e^{-1}.

(b) If $A\mathbf{x} = \lambda\mathbf{x}$, then $e^A\mathbf{x} = \lim\limits_{k\to\infty} [I + A + A^2/2! + \cdots + A^k/k!]\mathbf{x} = \lim\limits_{k\to\infty} [\mathbf{x} + \lambda\mathbf{x} + \lambda^2\mathbf{x}/2! + \cdots + \lambda^k\mathbf{x}/k!] = e^\lambda\mathbf{x}$.

(c) If $A = P\,\mathrm{diag}\,[\lambda_j]\,P^{-1}$, then by Corollary 1, $\det e^A = \det\{P\,\mathrm{diag}\,[e^{\lambda_j}]\,P^{-1}\} = \det\{\mathrm{diag}\,[e^{\lambda_j}]\} = e^{\lambda_1}\cdots e^{\lambda_k} = e^{\mathrm{trace}A}$. For a 2 x 2 matrix A with repeated eigenvalues λ, we have $\det e^A = \det\begin{bmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{bmatrix} = e^{2\lambda} = e^{\mathrm{trace}A}$; and for a 2 x 2 matrix A with complex eigenvalues, $\lambda = a \pm ib$, we have $\det e^A = \det\begin{bmatrix} e^a\cos b & -e^a\sin b \\ e^a\sin b & e^a\cos b \end{bmatrix} = e^{2a} = e^{\mathrm{trace}A}$ (since the trace $A = \lambda_1 + \lambda_2 = (a + ib) + (a - ib) = 2a$ in this case).

7. (a) $e^A = \mathrm{diag}[e, e^2, e^3]$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = N + S$ and $NS = SN$ so that by Proposition 2,

$$e^A = \mathrm{diag}\,[e, e^2, e^2]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}$$ since $N^2 = 0$ implies that $e^N = I + N$.

(c) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = N + S$ and $NS = SN$ so that by Proposition 2

$e^A = e^S e^N = e^2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}$ since $N^3 = 0$ implies that $e^N = I + N + N^2/2$.

8. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ we have $AB = 0 \neq BA = B$, $e^{A+B} = \begin{bmatrix} e & 0 \\ e-1 & 1 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} e & 0 \\ 1 & 1 \end{bmatrix}$.

9. If $T(\mathbf{x}) \in E$ for all $\mathbf{x} \in E$, then by induction $T^2(\mathbf{x}) \in E, \cdots, T^k(\mathbf{x}) \in E$ and therefore $e^T(\mathbf{x}) = \lim\limits_{k\to\infty} [I + T + \cdots + T^k/k!]\mathbf{x} = \lim\limits_{k\to\infty} [\mathbf{x} + T(\mathbf{x}) + \cdots + \frac{T^k(\mathbf{x})}{k!}] \in E$ since any subspace E of $\mathbf{R}^n$ is complete and since $\mathbf{x}_k = \mathbf{x} + T(\mathbf{x}) + \cdots + T^k(\mathbf{x})/k!$ is a Cauchy sequence in E.

PROBLEM SET 1.4

1. (a) $\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$

(b) $\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} x_0 e^{\lambda t} + y_0 te^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$

(c) $\mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0 = e^{at} \begin{bmatrix} x_0 \cos bt - y_0 \sin bt \\ x_0 \sin bt + y_0 \cos bt \end{bmatrix}$

2. $\mathbf{x}(t) = e^{-t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mathbf{x}_0$

3. (a) $\lambda_1 = 2, \lambda_2 = 4, \mathbf{v}_1 = (1, -1)^T, \mathbf{v}_2 = (1, 1)^T, \mathbf{x}(t) = P\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} P^{-1}\mathbf{x}_0 =$

$$1/2\begin{bmatrix} e^{2t}+e^{4t} & e^{4t}-e^{2t} \\ e^{4t}-e^{2t} & e^{4t}+e^{2t} \end{bmatrix}\mathbf{x}_0 = e^{3t}\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(b) $\lambda_1 = 4, \lambda_2 = -2, \mathbf{v}_1 = (1, 1)^T, \mathbf{v}_2 = (1, -1)^T, \mathbf{x}(t) = P\begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} P^{-1} =$

$$1/2\begin{bmatrix} e^{4t}+e^{-2t} & e^{4t}-e^{-2t} \\ e^{4t}-e^{-2t} & e^{4t}+e^{2t} \end{bmatrix}\mathbf{x}_0 = e^{t}\begin{bmatrix} \cosh 3t & \sinh 3t \\ \sinh 3t & \cosh 3t \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

4. From Problem 2 in Problem Set 2, $\mathbf{x}(t) = e^{At}\mathbf{x}_0 = P \operatorname{diag}[e^{\lambda_i t}] P^{-1} =$

$$1/2\begin{bmatrix} 2e^{t} & 0 & 0 \\ 2e^{2t}-2e^{t} & 2e^{2t} & 0 \\ e^{t}-e^{-t} & 0 & 2e^{-t} \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } P^{-1} = 1/2\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

5. (a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = S + N$ where S and N commute. Thus,

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\mathbf{x}_0.$$

(b) $\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\mathbf{x}_0.$

(c) $\lambda_1 = 1, \lambda_2 = -1, \mathbf{v}_1 = (1, 1)^T, \mathbf{v}_2 = (-1, 1)^T, \mathbf{x}(t) = e^{At}\mathbf{x}_0 = P^{-1}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} P\mathbf{x}_0 =$

$$1/2\begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}\mathbf{x}_0 = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}\mathbf{x}_0 \text{ where } P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, P^{-1} = 1/2\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

(d) $A = -2I + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = S + N$ where S and N commute. Thus, $\mathbf{x}(t) = e^{At}\mathbf{x}_0 =$

$$e^{-2t}[I + Nt + N^2t^2/2]\mathbf{x}_0 = e^{-2t}\begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}\mathbf{x}_0.$$

6. Since $T(\mathbf{x}) \in E$ for all $\mathbf{x} \in E$ and since $T(\mathbf{x}) = A\mathbf{x}$, it follows that if $\mathbf{x}_0 \in E$ then $A\mathbf{x}_0 \in E$ and $tA\mathbf{x}_0 \in E$ since E is a linear subspace of $\mathbf{R}^n$. It then follows by induction that $(t^k/k!)A^k\mathbf{x}_0 \in E$ for all $k \in \mathbf{N}$. Therefore $\sum_{k=0}^{N} A^k t^k \mathbf{x}_0/k! \in E$ since E is a linear subspace of $\mathbf{R}^n$. Then since a closed subset of a complete metric space is complete, it follows that E is a complete normed linear space; i.e., every Cauchy sequence in E converges to a vector in E. (Cf. Theorem 3.11, p. 53 in [R].) Thus, for all $t \in \mathbf{R}$ $\lim_{N\to\infty} \sum_{k=0}^{N} A^k t^k \mathbf{x}_0/k! = e^{At}\mathbf{x}_0 \in E$. And therefore by the Fundamental Theorem for linear systems $\mathbf{x}(t) = e^{At}\mathbf{x}_0 \in E$ for all $t \in \mathbf{R}$.

7. Suppose that there is a $\lambda < 0$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{x}(t) = e^{At}\mathbf{v}$ is a solution of (1) with $\mathbf{x}(0) = \mathbf{v}$. But $e^{At}\mathbf{v} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\mathbf{v} = \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!}\mathbf{v} = e^{\lambda t}\mathbf{v}$ since, by induction, $A^k\mathbf{v} = \lambda^k\mathbf{v}$. Thus, $\lim_{t\to\infty} \mathbf{x}(t) = \lim_{t\to\infty} e^{At}\mathbf{v} = \lim_{t\to\infty} e^{\lambda t}\mathbf{v} = \mathbf{0}$ since $\lambda < 0$.

8. By the Fundamental Theorem for linear systems, the solution of $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is given by $\boldsymbol{\phi}(t, \mathbf{x}_0) = e^{At}\mathbf{x}_0$. Thus, for all $t \in \mathbf{R}$, $\lim_{\mathbf{y}\to\mathbf{x}_0} \mathbf{f}(t,\mathbf{y}) = \lim_{\mathbf{y}\to\mathbf{x}_0} e^{At}\mathbf{y} = e^{At} \lim_{\mathbf{y}\to\mathbf{x}_0} \mathbf{y} = e^{At}\mathbf{x}_0 = \boldsymbol{\phi}(t,\mathbf{x}_0)$.

PROBLEM SET 1.5

1. (a) $\delta = -2 < 0$ implies that (1) has a saddle at the origin.

   (b) $\delta = 8$, $\tau = 6$, $\tau^2 - 4\delta = 4 > 0$ implies that (1) has an unstable node at the origin.

   (c) $\delta = 2$, $\tau = 0$ implies that (1) has a center at the origin.

   (d) $\delta = 5$, $\tau = 4$, $\tau^2 - 4\delta = -4$ implies that (1) has an unstable focus at the origin.

   (e) $\delta = \lambda^2 + 2 > 0$, $\tau = 2\lambda$, $\tau^2 - 4\delta = -8 < 0$ implies that for $\lambda \neq 0$ (1) has a focus at the origin; it is stable if $\lambda < 0$ and unstable if $\lambda > 0$; and (1) has a center at the origin if $\lambda = 0$.

   (f) $\delta = \lambda^2 - 2$, $\tau = 2\lambda$, $\tau^2 - 4\delta = 8 > 0$ implies that (1) has a saddle at the origin if $|\lambda| < \sqrt{2}$; (1) has a node at the origin if $|\lambda| > \sqrt{2}$; it is stable if $\lambda < -\sqrt{2}$ and unstable if $\lambda > \sqrt{2}$; and (1) has a degenerate critical point at the origin if $|\lambda| = \sqrt{2}$.

2. (a) $x_1(t) = x_1(0)e^{3t}$, $x_2(t) = x_2(0)e^{3t}$. Cf. Problem 3 with a = 1 in Problem Set 1.1.

(b) $x_1(t) = x_1(0)e^{3t}$, $x_2(t) = x_2(0)e^{t}$. Cf. Problem 3 with a = 1/2 in Problem Set 1.1.

(c) $x_1(t) = x_1(0)e^{t}$, $x_2(t) = x_2(0)e^{3t}$. Cf. Problem 3 with a = 2 in Problem Set 1.1.

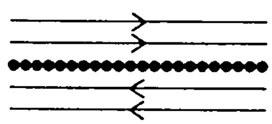
(d) $x_1(t) = [x_1(0) + x_2(0)t]e^{t}$, $x_2(t) = x_2(0)e^{t}$, which follows from $e^{At}\,\mathbf{x}_0 = e^{t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x_0$.

Cf. Figure 2.

3. $\delta = 2a + b^2 > 0$ iff $a > -b^2/2$ and $\tau = a + 2 < 0$ iff $a < -2$. Thus, the system $\dot{\mathbf{x}} = A\mathbf{x}$ has a sink at the origin iff $-b^2/2 < a < -2$.

4. (a) $x(t) = x_0 e^{\lambda t}$, $y(t) = y_0$. For $\lambda > 0$ cf. Problem 3 with a = 0 in Problem Set 1.1.

(b) $x(t) = x_0 + y_0 t$, $y(t) = y_0$.



(c) $x(t) = x_0$, $y(t) = y_0$; every point $\mathbf{x}_0 \in \mathbf{R}^2$ is a critical point.

5. The second-order differential equation can be written in the form of a linear system (1) with $A = \begin{bmatrix} 0 & -1 \\ b & a \end{bmatrix}$. If $b < 0$, the origin is a saddle; if $b > 0$ and $a^2 - 4b \geq 0$, the origin is a node which is stable if $a < 0$ and unstable if $a > 0$; if $b > 0$, $a^2 - 4b < 0$ and $a \neq 0$, the origin is a focus which is stable if $a < 0$ and unstable if $a > 0$; if $b > 0$ and $a = 0$, the origin is a center; and if $b = 0$, the origin is a degenerate critical point.

6. $x_1(t) = x_1(0)e^{t}$, $x_2(t) = x_1(0)e^{t} + [x_2(0) - x_1(0)]e^{2t}$; $\lambda_1 = 1$, $\lambda_2 = 2$, $\mathbf{v}_1 = (1, 1)^T$ and $\mathbf{v}_2 = (0, 1)^T$; the origin is an unstable node.

7. $\lambda_1 = (5 + \sqrt{33})/2$, $\lambda_2 = (5 - \sqrt{33})/2$, $\mathbf{v}_1 = (4, 3 + \sqrt{33})^T$, $\mathbf{v}_2 = (4, 3 - \sqrt{33})^T$; the separatrices are the four trajectories in $E^s \cup E^u$ and the origin.

8. Since $x_1(t) = x_1(0)\cos t - x_2(0)\sin t$ and $x_2(t) = x_1(0)\sin t + x_2(0)\cos t$, $r(t) = \sqrt{x_1^2(t) + x_2^2(t)} = \sqrt{x_1^2(0) + x_2^2(0)}$, a constant and $\theta(t) = \tan^{-1}[x_2(t)/x_1(t)] = \tan^{-1}[x_2(0)/x_1(0)] + t$; the origin is a center for this system.

9. Differentiating $r^2 = x_1^2 + x_2^2$ with respect to t leads to $2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$ or $\dot{r} = (x_1\dot{x}_1 + x_2\dot{x}_2)/r$ for $r \neq 0$. Differentiating $\theta = \tan^{-1}(x_2/x_1)$ with respect to t leads to $\dot{\theta} = (x_1\dot{x}_2 - x_2\dot{x}_1)/x_1^2[1 + (x_2/x_1)^2] = (x_1\dot{x}_2 - x_2\dot{x}_1)/r^2$ for $r \neq 0$. For the system in Problem 8 we easily obtain $\dot{r} = ar$ and $\dot{\theta} = b$ from these equations. These latter equations with the initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$ have the solution $r(t) = r_0 e^{at}$, $\theta(t) = \theta_0 + bt$. Thus for $a < 0$, $r(t) \to 0$ as $t \to \infty$ and for $b > 0$ (or $b < 0$), $\theta(t) \to \infty$ as $t \to \infty$ (or as $t \to -\infty$) as in Figure 3. And for $a = 0$, $r(t) = r_0$ while $\theta(t) \to \infty$ as $t \to \infty$ (or as $t \to -\infty$) for $b > 0$ (or $b < 0$) as in Figure 4.

PROBLEM SET 1.6

1. $\lambda = 2 \pm i$. For $\lambda = 2 + i$, $\mathbf{w} = \mathbf{u} + i\mathbf{v} = (1, 1)^T + i(1, 0)^T$,

$$P = [\mathbf{v}\ \mathbf{u}] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, P^{-1}AP = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

and the solution $\mathbf{x}(t) = Pe^{2t} R_t P^{-1} \mathbf{x}_0 = e^{2t} \begin{bmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{bmatrix} \mathbf{x}_0$

where $R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$.

2. $\lambda = 1 \pm i$, $\lambda_3 = -2$, $\mathbf{w} = (1 - i, -1, 0)^T$, $\mathbf{v}_3 = (0, 0, 1)^T$, $P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the solution $\mathbf{x}(t) = P \begin{bmatrix} e^t\cos t & -e^t\sin t & 0 \\ e^t\sin t & e^t\cos t & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} P^{-1}\mathbf{x}_0 =$

$$\begin{bmatrix} e^t(\cos t - \sin t) & -2e^t\sin t & 0 \\ e^t\sin t & e^t(\sin t + \cos t) & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \mathbf{x}_0.$$

(Also, see p. 122 in the appendix.)

3. $\lambda_1 = 1, \lambda = 2 \pm 3i$, $\mathbf{v}_1 = (-10, 3, 1)$, $\mathbf{w} = (0, i, 1)^T$, $P = \begin{bmatrix} -10 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} -1/10 & 0 & 0 \\ 3/10 & 1 & 0 \\ 1/10 & 0 & 1 \end{bmatrix}$ and the solution $\mathbf{x}(t) = P \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t}\cos 3t & -e^{2t}\sin 3t \\ 0 & e^{2t}\sin 3t & e^{2t}\cos 3t \end{bmatrix} P^{-1}\mathbf{x}_0 =$

$$\begin{bmatrix} e^t & 0 & 0 \\ (-3e^t + 3e^{2t}\cos 3t - e^{2t}\sin 3t)/10 & e^{2t}\cos 3t & -e^{2t}\sin 3t \\ (-e^t + 3e^{2t}\sin 3t + e^{2t}\cos 3t)/10 & e^{2t}\sin 3t & e^{2t}\cos 3t \end{bmatrix} \mathbf{x}_0.$$

4. $\lambda_1 = -1 + i$, $\lambda_3 = 1 + i$, $\mathbf{w}_1 = (1, -i, 0, 0)^T$, $\mathbf{w}_3 = (0, 0, 1 - i, -1)^T$,

$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and the solution $\mathbf{x}(t) =$

$$P \begin{bmatrix} e^{-t}R_t & 0 \\ 0 & e^t R_t \end{bmatrix} P^{-1}\mathbf{x}_0 = \begin{bmatrix} e^{-t}\cos t & -e^{-t}\sin t & 0 & 0 \\ e^{-t}\sin t & e^{-t}\cos t & 0 & 0 \\ 0 & 0 & e^t(\cos t - \sin t) & -2e^t \sin t \\ 0 & 0 & e^t \sin t & e^t(\sin t + \cos t) \end{bmatrix} \mathbf{x}_0.$$

PROBLEM SET 1.7

1. (a) $\lambda_1 = \lambda_2 = 1$; $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = S + N$ where S and N commute and $N^2 = 0$;

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{St}e^{Nt}\mathbf{x}_0 = e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \mathbf{x}_0.$$

(b) $\lambda_1 = \lambda_2 = 2$; $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = S + N$ where S and N commute and $N^2 = 0$;

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{St}e^{Nt}\mathbf{x}_0 = e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix} \mathbf{x}_0.$$

(c) $\lambda_1 = 1, 1$; $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = S + N$; $\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{St}e^{Nt}\mathbf{x}_0 = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$.

(d) $\lambda_1 = 1, \lambda_2 = -1$; $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = S + N$, but S and N do not commute; therefore,

we must find $\mathbf{v}_1 = (1, 0)^T$, $\mathbf{v}_2 = (1, -2)^T$, $P = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$, $P^{-1} = 1/2 \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and

$$\mathbf{x}(t) = P \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} P^{-1}\mathbf{x}_0 = 1/2 \begin{bmatrix} 2e^t & e^t - e^{-t} \\ 0 & 2e^{-t} \end{bmatrix} \mathbf{x}_0.$$

2. (a) $\lambda_1 = \lambda_2 = \lambda_3 = 1$; $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} = S + N$ where S and N

commute and $N^3 = 0$; therefore $\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^t e^{Nt}\mathbf{x}_0 = e^t \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ 3t + 2t^2 & 2t & 1 \end{bmatrix} \mathbf{x}_0$.

(b) $\lambda_1 = \lambda_2 = -1, \lambda_3 = 1$ and we must compute the generalized eigenvectors; $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$ satisfying $(A - \lambda_1 I)^2 \mathbf{v}_2 = \mathbf{0}$, and $\mathbf{v}_3 = (0, 2, 1)^T$; $S = P \operatorname{diag}[-1, -1, 1] P^{-1} =$

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, $N = A - S = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, S and N commute, $N^2 = 0$ and $\mathbf{x}(t) = e^{At}\mathbf{x}_0 =$

$$e^{St}[I + Nt]\,\mathbf{x}_0 = P \operatorname{diag}[e^{-t}, e^{-t}, e^t]\, P^{-1}\,[I + Nt]\mathbf{x}_0 = \begin{bmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & 2(e^t - e^{-t}) \\ 0 & 0 & e^t \end{bmatrix} \mathbf{x}_0.$$

(c) $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2$ and in this case there is a basis of eigenvectors; $\mathbf{v}_1 = (1, 1, -1)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$, $\mathbf{v}_3 = (0, 0, 1)^T$; $A = P \operatorname{diag}[1, 2, 2] P^{-1}$ and $\mathbf{x}(t) = e^{At}\mathbf{x}_0 =$

$$P \operatorname{diag}[e^t, e^{2t}, e^{2t}]P^{-1}\mathbf{x}_0 = \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ e^{2t} - e^t & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(d) $\lambda_1 = \lambda_2 = \lambda_3 = 2$; $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = S + N$ where S and N commute and $N^3 = 0$; therefore, $\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{2t}e^{Nt}\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1 & t & t+t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}_0.$

3. (a) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$; $A = N$ is nilpotent with $A^3 = 0$ and

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 - t^2 2 & t^2/2 & t \\ t & -t^2/2 & 1 + t^2/2 & t \\ 0 & -t & t & 1 \end{bmatrix}\mathbf{x}_0.$$

(b) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$; $A = 2I + N$ where $N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $N^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

$N^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $N^4 = 0$ and $\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{2t}e^{Nt}\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2/2 & t & 1 & 0 \\ t^3/6 & t^2 2 & t & 1 \end{bmatrix}\mathbf{x}_0.$

(c) $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 10$ and there is a basis of eigenvectors, $\mathbf{v}_1 = (1, -1, 0, 0)^T$,

$\mathbf{v}_2 = (1, 0, -1, 0)^T$, $\mathbf{v}_3 = (1, 0, 0, -1)^T$, $\mathbf{v}_4 = (1, 2, 3, 4)^T$, $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 4 \end{bmatrix}$,

$P^{-1} = 1/10 \begin{bmatrix} 2 & -8 & 2 & 2 \\ 3 & 3 & -7 & 3 \\ 4 & 4 & 4 & -6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = P\,\mathrm{diag}\,[1, 1, 1, e^{10t}]\,P^{-1} =$

$$1/10 \begin{bmatrix} 9+e^{10t} & -1+e^{10t} & -1+e^{10t} & -1+e^{10t} \\ -2+2e^{10t} & 8+2e^{10t} & -2+2e^{10t} & -2+2e^{10t} \\ -3+3e^{10t} & -3+3e^{10t} & 7+3e^{10t} & -3+3e^{10t} \\ -4+4e^{10t} & -4+4e^{10t} & -4+4e^{10t} & 6+4e^{10t} \end{bmatrix} \mathbf{x}_0.$$

(d) $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 1 + i$, $\lambda_4 = 1 - i$, $\mathbf{v}_1 = (1, 1, 0, 0)^T$, $\mathbf{v}_2 = (1, -1, 0, 0)$,

$\mathbf{w}_3 = (0, 0, i, 1)$, $P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$$P \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^t\cos t & -e^t\sin t \\ 0 & 0 & e^t\sin t & e^t\cos t \end{bmatrix} P^{-1}\,\mathbf{x}_0 = \begin{bmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & e^t\cos t & -e^t\sin t \\ 0 & 0 & e^t\sin t & e^t\cos t \end{bmatrix} \mathbf{x}_0.$$

(e) $\lambda_1 = \lambda_2 = 1 + i$ and the eigenvectors $\mathbf{w}_1 = (i, 1, 0, 0)^T$, $\mathbf{w}_2 = (0, 0, i, 1)^T$ lead to

$$P = I,\ A = S = \mathrm{diag}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{x}(t) = e^t \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} \mathbf{x}_0.$$

(f) $\lambda_1 = \lambda_2 = 1 + i$; the eigenvector $\mathbf{w}_1 = (i, 1, 0, 0)^T$ and the generalized eigenvector

$\mathbf{w}_2 = (i, 1, i, 1)^T$, satisfying $(A - \lambda_1 I)\,\mathbf{w}_2 = \mathbf{w}_1$, lead to $P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $S = P\,\text{diag}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} P^{-1} = \text{diag}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and

therefore $N = A - S = \begin{bmatrix} 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ 0 & 0 \end{bmatrix}$ with $SN = NS$ and $N^2 = 0$; $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$e^t\,P\,\text{diag}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} P^{-1}\,[I + Nt]\,\mathbf{x}_0 = e^t \begin{bmatrix} \cos t & -\sin t & t\cos t & -t\sin t \\ \sin t & \cos t & t\sin t & t\cos t \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} \mathbf{x}_0.$

4. (a) $\lambda_1 = \lambda_2 = 2$, $P_1 = A - 2I = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $r_1(t) = e^{2t}$, $r_2(t) = te^{2t}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$[r_1(t)I + r_2(t)P_1]\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix} \mathbf{x}_0.$

(b) $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$, $r_1(t) = e^t$, $r_2(t) = e^{2t} - e^t$, $r_3(t) = te^{2t} - e^{2t} + e^t$, $P_1 = A - I =$

$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $P_2 = (A - I)\,(A - 2I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 =$

$[r_1(t)I + r_2(t)P_1 + r_3(t)P_2]\,\mathbf{x}_0 = \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{bmatrix} \mathbf{x}_0.$

(c) $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$, $r_1(t) = e^t$, $r_2(t) = e^{2t} - e^t$, $P_1 = A - I$, $P_2 = 0$,

$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = [r_1(t)I + r_2(t)P_1]\,\mathbf{x}_0 = \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ e^{2t} - e^t & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$

(d) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$, $r_1(t) = e^{2t}$, $r_2(t) = te^{2t}$, $r_3(t) = t^2 e^{2t}/2$, $r_4(t) = t^3 e^{2t}/6$,

$P_1 = N$, $P_2 = N^2$, $P_3 = N^3$, as in Problem 3(b); thus, $\mathbf{x}(t) = e^{At}\mathbf{x}_0 =$

$$[r_1(t)I + r_2(t)N + r_3(t)N^2 + r_4(t)N^3] = e^{2t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2/2 & t & 1 & 0 \\ t^3/6 & t^2/2 & t & 1 \end{bmatrix}\mathbf{x}_0.$$

PROBLEM SET 1.8

1. (a), (b), (d), (f) and (h) are already in Jordan canonical form.

(c), (e) $\lambda_1 = 1, \lambda_2 = -1$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, (g) $\lambda_1 = 2, \lambda_2 \doteq 0$ and $J = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$,

(i) $\lambda_1 = \lambda_2 = 1$, $\delta_1 = 1$, $\delta_2 = 2$, $\nu_1 = 0$, $\nu_2 = 1$ and $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

2. (a), (b), (c), (d) are already in Jordan canonical form.

(e) and (f) $\lambda_1 = \lambda_2 = 1$, $\lambda_2 = -1$, $\delta_1 = 2$ and $J = \text{diag}\,[1, 1, -1]$.

3. (a)

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 4$

$\nu_1 = 4$, $\nu_2 = \nu_3 = \nu_4 = 0$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 3$, $\delta_2 = \delta_3 = \delta_4 = 4$

$\nu_1 = 2$, $\nu_2 = 1$, $\nu_3 = \nu_4 = 0$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 2$, $\delta_2 = 3$, $\delta_3 = \delta_4 = 4$

$\nu_1 = 1$, $\nu_2 = 0$, $\nu_3 = 1$, $\nu_4 = 0$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 1$, $\delta_2 = 2$, $\delta_3 = 3$, $\delta_4 = 4$

$\nu_1 = \nu_2 = \nu_3 = 0$, $\nu_4 = 1$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$\delta_1 = 2$, $\delta_2 = \delta_3 = \delta_4 = 4$

$\nu_1 = \nu_3 = \nu_4 = 0$, $\nu_2 = 2$

(b) $$\mathbf{x}(t) = e^{\lambda t}\,\mathbf{x}_0,\ \mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0,\ \mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & t^2/2 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0,$$

$$\mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0,\ \mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,\mathbf{x}_0.$$

4. (a) $$\begin{bmatrix} a_1 & -b_1 & 0 & 0 \\ b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & -b_2 \\ 0 & 0 & b_2 & a_2 \end{bmatrix},\ \begin{bmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix},\ \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix},\ \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

(b) $$\mathbf{x}(t) = P \begin{bmatrix} e^{a_1 t} R_{b_1 t} & 0 \\ 0 & e^{a_2 t} R_{b_2 t} \end{bmatrix} P^{-1}\,\mathbf{x}_0,\ \mathbf{x}(t) = e^{at} P \begin{bmatrix} R_{bt} & tR_{bt} \\ 0 & R_{bt} \end{bmatrix} P^{-1}\,\mathbf{x}_0,$$

$$\mathbf{x}(t) = P \begin{bmatrix} e^{at} R_{bt} & 0 \\ 0 & \begin{matrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{matrix} \end{bmatrix} P^{-1}\,\mathbf{x}_0,\ \mathbf{x}(t) = P \begin{bmatrix} e^{at} R_{bt} & 0 \\ 0 & \begin{matrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{matrix} \end{bmatrix} P^{-1}\,\mathbf{x}_0.$$

5. (a) $$\begin{bmatrix} \lambda &&&& \\ & \lambda &&& \\ && \lambda && \\ &&& \lambda & \\ &&&& \lambda \end{bmatrix},\quad \begin{bmatrix} \lambda & 1 &&& \\ & \lambda &&& \\ && \lambda && \\ &&& \lambda & \\ &&&& \lambda \end{bmatrix},\quad \begin{bmatrix} \lambda & 1 &&& \\ & \lambda & 1 && \\ && \lambda && \\ &&& \lambda & \\ &&&& \lambda \end{bmatrix},$$

$\delta_1 = 5$ $\qquad$ $\delta_1 = 4, \delta_2 = 5$ $\qquad$ $\delta_1 = 3, \delta_2 = 4, \delta_3 = 5$

$$\begin{bmatrix} \lambda & 1 &&& \\ & \lambda & 1 && \\ && \lambda & 1 & \\ &&& \lambda & \\ &&&& \lambda \end{bmatrix},\quad \begin{bmatrix} \lambda & 1 &&& \\ & \lambda & 1 && \\ && \lambda & 1 & \\ &&& \lambda & 1 \\ &&&& \lambda \end{bmatrix},$$

$\delta_1 = 2, \delta_2 = 3, \delta_3 = 4, \delta_4 = 5$ $\qquad$ $\delta_1 = 1, \delta_2 = 2, \delta_3 = 3, \delta_4 = 4, \delta_5 = 5$

$$\begin{bmatrix} \lambda & 1 &&& \\ & \lambda &&& \\ && \lambda & 1 & \\ &&& \lambda & \\ &&&& \lambda \end{bmatrix},\quad \begin{bmatrix} \lambda & 1 &&& \\ & \lambda & 1 && \\ && \lambda && \\ &&& \lambda & 1 \\ &&&& \lambda \end{bmatrix}$$

$\delta_1 = 3, \delta_2 = 5, \delta_3 = \delta_4 = \delta_5 = 0$ $\qquad$ $\delta_1 = 2, \delta_2 = 4, \delta_3 = 5, \delta_4 = \delta_5 = 0.$

(b) For example, in the fifth case we have $\mathbf{x}(t) = e^{\lambda t} P \begin{bmatrix} 1 & t & t^2/2 & t^3/6 & t^4/24 \\ 0 & 1 & t & t^2/2 & t^3/6 \\ 0 & 0 & 1 & t & t^2/2 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1} \mathbf{x}_0$.

6. (a) $J = \mathrm{diag}[1, 2, 3]$.

(b) $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2, \delta_1 = 2$, and $J = \mathrm{diag}\ [1, 2, 2]$.

(c) $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2, \delta_1 = 1, \delta_2 = 2$ and $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(d) $\lambda_1 = \lambda_2 = \lambda_3 = 2, \delta_1 = 1, \delta_2 = 2, \delta_3 = 3$ and $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(e) $J = \mathrm{diag}\ [1, 2, 3, 4]$.

(f) $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 2, \delta_1 = 2, \delta_2 = 3$ (for $\lambda = 2$) and $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(g) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2, \delta_1 = 2, \delta_2 = 3, \delta_3 = 4$ and as in Problem 3(a), $J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(h) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2, \delta_1 = 1, \delta_2 = 2, \delta_3 = 3, \delta_4 = 4$ and see Problem 3(a) solution.

The solutions, which follow from $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = P\,e^{Jt}\,P^{-1}\,\mathbf{x}_0$:

(a) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \text{diag}\,[e^t, e^{2t}, e^{3t}] \begin{bmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \\ 3/2 & 2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} e^t & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 \\ \dfrac{e^t}{2} - 2e^{2t} + \dfrac{3e^{3t}}{2} & 2e^{3t} - 2e^{2t} & e^{3t} \end{bmatrix}.$$

(b) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{diag}[e^t, e^{2t}, e^{2t}] \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ e^{2t} - e^t & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(c) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0 =$$

$$\begin{bmatrix} e^t & e^{2t} - e^t & e^{2t} - e^t + te^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(d) $$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & t^2e^{2t}/2 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0 =$$

$$e^{2t} \begin{bmatrix} 1 & t & 2t + t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0.$$

(f) $\quad \mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}_0 =$

$$\begin{bmatrix} e^t & 0 & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 & 0 \\ e^{2t} - e^t & 0 & e^{2t} & 0 \\ te^{2t} & te^{2t} & 0 & e^{2t} \end{bmatrix} \mathbf{x}_0.$$

(h) $\quad \mathbf{x}(t) = e^{At}\,\mathbf{x}_0 = e^{2t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 12 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0 =$

$$e^{2t} \begin{bmatrix} 1 & t & 4t + t^2/2 & 3t^2/2 + t^3/6 \\ 0 & 1 & t & t^2/2 - t \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0.$$

(Also, see p. 122 in the appendix.)

7. If $Q = \mathrm{diag}\,[1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{m-1}]$, then $Q^{-1} = \mathrm{diag}\left[1, \frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}, \ldots, \frac{1}{\varepsilon^{m-1}}\right]$ and $Q^{-1}BQ =$

$Q^{-1}(\lambda I + N)Q = \lambda I + Q^{-1}NQ$ where $Q^{-1}NQ =$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\varepsilon} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\varepsilon^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\varepsilon^{m-1}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \varepsilon^{m-1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1/\varepsilon & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1/\varepsilon^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1/\varepsilon^{m-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \varepsilon^{m-1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & \varepsilon & 0 & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & 0 & \varepsilon & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \varepsilon N.$$

8. The eigenvalues of a nilpotent matrix are all equal to zero. (This follows from the fact that any nilpotent matrix is linearly equivalent to a matrix with blocks of the form N^k along the diagonal where each N^k has the form of one of the matrices shown on the page following the statement of the theorem in this section.)

9. By the corollary in this section, each coordinate of the solution $\mathbf{x}(t)$ of the initial value problem (4) is a linear combination of functions of the form $t^k e^{at} \cos bt$ or $t^k e^{at} \sin bt$ where k is a non-negative integer and the coefficients depend on the initial conditions $\mathbf{x}_0$. But if all of the eigenvalues of A have a negative real part, then $a = \mathrm{Re}(\lambda) < 0$ in these functions and since for all $a < 0$ and all integers k, $t^k e^{at} \to 0$ as $t \to \infty$ (and since $|\cos bt| \leq 1$ and $|\sin bt| \leq 1$), it follows that for all $\mathbf{x}_0 \in \mathbf{R}^n$ each coordinate of $\mathbf{x}(t)$ approaches zero as $t \to \infty$; i.e., $\mathbf{x}(t) \to 0$ as $t \to \infty$.

10. If the elementary blocks in the Jordan form of A have the form $B = \mathrm{diag}\,[\lambda, \ldots, \lambda]$ or $B = \mathrm{diag}\,[D, \ldots, D]$ where D is a 2 x 2 matrix of the form in the theorem stated in this section, then each coordinate in the solution $\mathbf{x}(t)$ of the initial value problem (4) will be a linear combination of functions of the form $e^{\lambda t}$, $e^{at} \cos bt$ or $e^{at} \sin bt$. Furthermore, if all of the eigenvalues of A have non-positive real part, i.e., if $\lambda \leq 0$ and $a \leq 0$ in the above forms, then each of the coordinates of $\mathbf{x}(t)$ are bounded by constants (depending on $\mathbf{x}_0 \in \mathbf{R}^n$) for all $t \geq 0$ and therefore for each $\mathbf{x}_0 \in \mathbf{R}^n$, there exists a positive constant M such that $|\mathbf{x}(t)| \leq M$ for all $t \geq 0$.

11. Example 4 in Section 1.7 has $|\lambda| = 1$ and yet the functions $t\cos t$ and $t\sin t$ are not bounded as $t \to \infty$ (or as $t \to -\infty$). In particular, the solution with $\mathbf{x}_0 = (1, 0, 0, 0)^T$ has $|\mathbf{x}(t)| = \sqrt{1 + t^2 + \sin^2 t + t\sin 2t} \geq |t - 1|$ and therefore $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. Also, note that any solution of Example 4 in Section 1.7 with $\mathbf{x}_0 \in \text{Span}\,\{(0, 0, 1, 0)^T, (0, 0, 0, 1)^T\}$ remains bounded for all $t \in \mathbf{R}$.

12. Since this problem is closely related to Problem 5 in Set 9, we shall use the notation and theorems of the next section and do both of these problems at the same time; the corollary in this section tells us that the components of $\mathbf{x}(t)$ are linear combinations of functions of the form $t^k e^{at}\cos bt$ or $t^k e^{at}\sin bt$ with $\lambda = a + ib$ and $0 \leq k \leq n - 1$.

(a) This case occurs iff $\mathbf{x}_0 \in E^s \sim \{\mathbf{0}\}$.

(b) This case occurs iff $\mathbf{x}_0 \in E^u \sim \{\mathbf{0}\}$.

(c) This case occurs if $\mathbf{x}_0 \in E^c \sim \{\mathbf{0}\}$ and A is semisimple. (It may also occur if $\mathbf{x}_0 \in E^c \sim \{\mathbf{0}\}$ even if A is not semisimple as in Example 4 in Section 1.7.) That $|\mathbf{x}(t)| \geq m$ follows from the fact that $\mathbf{x}(t)$ is a periodic solution which does not intersect the critical point at the origin.

(d) This case occurs if $E^s \neq \{\mathbf{0}\}$, $E^u \neq \{\mathbf{0}\}$ and $\mathbf{x}_0 \in E^u \oplus E^s \oplus E^c \sim (E^u \cup E^s \cup E^c)$. (It may also occur for certain $\mathbf{x}_0 \in E^c \sim \{\mathbf{0}\}$ as in Example 4 in Section 1.7; cf. Problem 11 above.)

(e) This case occurs if $E^u \neq \{\mathbf{0}\}$, $E^c \neq \{\mathbf{0}\}$ and $\mathbf{x}_0 \in E^u \oplus E^c \sim (E^u \cup E^c)$.

(f) This case occurs if $E^s \neq \{\mathbf{0}\}$, $E^c \neq \{\mathbf{0}\}$ and $\mathbf{x}_0 \in E^s \oplus E^c \sim (E^s \cup E^c)$.

Furthermore, these are the only possible types of behavior that can occur as $t \to \pm\infty$ according to the corollary in this section.

PROBLEM SET 1.9

1. (a) $E^s = \text{Span}\,\{(0, 1)^T\}$, $E^u = \text{Span}\,\{(1, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   (b) $E^s = E^u = \{\mathbf{0}\}$, $E^c = \mathbf{R}^2$.

   (c) $E^s = E^c = \{\mathbf{0}\}$, $E^u = \mathbf{R}^2$.

   (d) $E^s = \text{Span}\,\{(1, 0)^T\}$, $E^u = \text{Span}\,\{(1, -1)^T\}$, $E^c = \{\mathbf{0}\}$.

   (e, f) $E^s = \text{Span}\,\{(1\ -1)^T\}$, $E^u = \text{Span}\,\{(1, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   (g) $E^s = \text{Span}\,\{(0, 1)^T\}$, $E^u = \{\mathbf{0}\}$, $E^c = \text{Span}\,\{(1, 0)^T\}$.

   (h) $E^s = E^u = \{\mathbf{0}\}$, $E^c = \mathbf{R}^2$.

   (i) $E^s = \mathbf{R}^2$, $E^s = E^c = \{\mathbf{0}\}$.

   The flow e^{At} is hyperbolic exactly when $E^c = \{\mathbf{0}\}$.

2. (a) $E^s = \text{Span}\,\{(1, 0, 0)^T, (0, 1, 0)^T\}$, $E^u = \text{Span}\,\{(0, 0, 1)^T\}$, $E^c = \{\mathbf{0}\}$.

   (b) $E^s = \text{Span}\,\{(0, 0, 1)^T\}$, $E^u = \{\mathbf{0}\}$, $E^c = \text{Span}\,\{(1, 0, 0)^T, (0, 1, 0)^T\}$.

   (c) $E^s = \text{Span}\,\{(1, 0, 0)^T, (0, 0, 1)^T\}$, $E^u = \text{Span}\,\{1, -1, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   (d) $E^s = \text{Span}\,\{(1, -1, 0)^T, (0, 0, 1)^T\}$, $E^u = \text{Span}\,\{(1, 0, 0)^T\}$, $E^c = \{\mathbf{0}\}$.

   The flow is hyperbolic in (a, c, d).

3. $\lambda = \pm 2i$, $\lambda_3 = 6$, $\mathbf{w}_1 = \mathbf{u}_1 + i\mathbf{v}_1 = (10, 0, -3)^T + i(0, 10, -1)^T$, $\mathbf{v}_3 = (0, 0, 1)^T$.

   $E^s = \{\mathbf{0}\}$, $E^u = \text{Span}\,\{(0, 0, 1)^T\}$, $E^c = \text{Span}\,\{(0, 10, -1)^T, (10, 0, -3)^T\}$.

$$\mathbf{x}(t) = \frac{1}{10}\begin{bmatrix} 0 & 10 & 0 \\ 10 & 0 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & e^{6t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \mathbf{x}_0 =$$

$$\frac{1}{10}\begin{bmatrix} 10\cos 2t & 10\sin 2t & 0 \\ -10\sin 2t & 10\cos 2t & 0 \\ \sin 2t - 3\cos 2t + 3e^{6t} & -\cos 2t - 3\sin 2t + e^{6t} & e^{6t} \end{bmatrix} \mathbf{x}_0.$$

For $\mathbf{x}_0 = (0, 0, c)^T \in E^u$, $\mathbf{x}(t) = (0, 0, e^{6t}c)^T \in E^u$; for $\mathbf{x}_0 \in E^c$, i.e., for $\mathbf{x}_0 = (10a, 10b, -3a - b)^T$, $\mathbf{x}(t) = (10(a\cos 2t + b\sin 2t), 10(b\cos 2t - a\sin 2t), -3(a\cos 2t + b\sin 2t) - (b\cos 2t - a\sin 2t))^T \in E^c$; and for $\mathbf{x}_0 = \mathbf{0} \in E^s$, $\mathbf{x}(t) = \mathbf{0} \in E^s$.

4. (a) $E^s = \text{Span}\{(1, 0, 0,)^T, (0, 1, 0)^T\}$, $E^u = \text{Span}\{(0, 2, 1)^T\}$, $E^c = \{\mathbf{0}\}$.

(b) $E^s = E^c = \{\mathbf{0}\}$, $E^u = \mathbf{R}^3$.

5. See Problem 12 in Set 8.

6. If $L : ax_1 + bx_2 = 0$ is an invariant line for the system (1), then for $\mathbf{x}_0 \in L$; $e^{At}\mathbf{x}_0 \in L$ for all $t \in \mathbf{R}$. But $\mathbf{x}_0 = (x_1, x_2)^T \in L$ and $\mathbf{x}_0 \neq \mathbf{0}$ implies that $\mathbf{x}_0 = k_1(-b, a)^T$ with $k_1 \neq 0$. And then $e^{At}\mathbf{x}_0 \in L$ for all $t \in \mathbf{R}$ implies that for all $t \in \mathbf{R}$ $e^{At}k_1(-b, a)^T = k_2(-b, a)^T$, and in particular that $e^A\mathbf{v} = k\mathbf{v}$ with $k = k_2/k_1$ and $\mathbf{v} = (-b, a)^T$. As in Section 1.5, if (1) has an invariant line then $A = PBP^{-1}$ and either $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ or $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. In the first case, if $\lambda \neq \mu$ it follows that either $k = e^\lambda$ and $P^{-1}\mathbf{v} = (1, 0)^T$ is an eigenvector of B, i.e., $\mathbf{v}$ is an eigenvector of A, or $k = e^\mu$ and $P^{-1}\mathbf{v} = (0, 1)^T$ is an eigenvector of B, i.e., $\mathbf{v}$ is an eigenvector of A. Also, in the first case if $\lambda = \mu$, then any vector $\mathbf{v} \in \mathbf{R}^2$ is an eigenvector of A and, in particular, $\mathbf{v} = (-b, a)^T$ is an eigenvector of A. In the second case $k = e^\lambda$ and $P^{-1}\mathbf{v} = (1, 0)^T$ is an eigenvector of B, i.e., $\mathbf{v}$ is an eigenvector of A and we are done. (The converse of Problem 6, that if $\mathbf{v} = (v_1, v_2)^T$ is an eigenvector of A, then $v_2x_1 - v_1x_2 = 0$ is an invariant line of (1) follows immediately from Problem 6 in Set 3.)

PROBLEM SET 1.10

1. Let $\Phi(t)$ be a fundamental matrix solution of (2) and let $\mathbf{x}(t) = \Phi(t)\mathbf{c}(t)$. Then $\mathbf{c}(0) = \Phi^{-1}(0)\mathbf{x}_0$ and $\dot{\mathbf{x}}(t) = \dot{\Phi}(t)\mathbf{c}(t) + \Phi(t)\dot{\mathbf{c}}(t) = A\Phi(t)\mathbf{c}(t) + \Phi(t)\dot{\mathbf{c}}(t)$ while $A\mathbf{x}(t) + \mathbf{b}(t) = A\Phi(t)\mathbf{c}(t) + \mathbf{b}(t)$. It then follows from (1) that $\Phi(t)\dot{\mathbf{c}}(t) = \mathbf{b}(t)$, i.e., that $\mathbf{c}(t) = \mathbf{c}(0) + \int_0^t \Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau = \Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau$. Thus $\mathbf{x}(t) = \Phi(t)\mathbf{c}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau$ which is equation (3).

2. $\lambda_1 = 1, \lambda_2 = -1, \mathbf{v}_1 = (1, 0)^T, \mathbf{v}_2 = (1, -2)^T$ and a fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ is given by $\Phi(t) = e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} e^t & \left(e^t - e^{-t}\right)/2 \\ 0 & e^{-t} \end{bmatrix}$.

Note that $\Phi^{-1}(t) = \Phi(-t)$ and then

$$\mathbf{x}(t) = \begin{bmatrix} e^t & \dfrac{\left(e^t - e^{-t}\right)}{2} \\ 0 & e^{-t} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{bmatrix} e^t & \dfrac{\left(e^t - e^{-t}\right)}{2} \\ 0 & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-\tau} & \dfrac{\left(e^{-\tau} - e^{\tau}\right)}{2} \\ 0 & e^{\tau} \end{bmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} d\tau =$$

$$\begin{pmatrix} -t - 2 + \dfrac{5}{2} e^t + \dfrac{1}{2} e^{-t} \\ 1 - e^{-t} \end{pmatrix}.$$

3. $\dot{\Phi}(t) = \begin{bmatrix} -2e^{-2t}\cos t - e^{-2t}\sin t & -\cos t \\ -2e^{-2t}\sin t + e^{-2t}\cos t & -\sin t \end{bmatrix} = A(t)\Phi(t)$, $\Phi^{-1}(t) = e^{2t}\begin{bmatrix} \cos t & \sin t \\ -e^{-2t}\sin t & e^{-2t}\cos t \end{bmatrix}$,

$\Phi^{-1}(0) = I$, and $\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\int_0^t e^{2\tau}\begin{bmatrix} \cos\tau & \sin\tau \\ -2e^{-2\tau}\sin\tau & e^{-2\tau}\cos\tau \end{bmatrix}\begin{pmatrix} 1 \\ e^{-2\tau} \end{pmatrix} d\tau =$

$$\Phi(t)\left[\mathbf{x}_0 + \begin{pmatrix} \dfrac{e^{2t}}{5}(2\cos t + \sin t) - \cos t + \dfrac{3}{5} \\ \dfrac{-e^{-2t}}{5}(2\cos t - \sin t) + \cos t - \dfrac{3}{5} \end{pmatrix}\right] =$$

$$\Phi(t)\mathbf{x}_0 + \frac{1}{5}\begin{pmatrix} 2\cos^2 t - 4\sin t\cos t + 3\sin t + e^{-2t}\left(-5\cos^2 t + 2\sin t\cos t - \sin^2 t + 3\cos t\right) \\ \sin^2 t + 2\sin t\cos t + 5\cos^2 t - 3\cos t + e^{-2t}\left(-2\cos^2 t - 4\sin t\cos t + 3\sin t\right) \end{pmatrix}.$$

2. NONLINEAR SYSTEMS: LOCAL THEORY

PROBLEM SET 2.1

1. $Df(x) = \begin{bmatrix} 1+x_2^2 & 2x_1x_2 \\ 2x_1 & -1+2x_2 \end{bmatrix}$, $Df(\mathbf{0}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Df(0, 1) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$,

$D^2f(0, 1)(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 2x_1y_2 + 2x_2y_1 \\ 2x_1y_1 + 2x_2y_2 \end{pmatrix}$.

(Also, see p. 123 in the appendix.)

2. (a) $E = \mathbf{R}^2 \sim \{\mathbf{0}\}$. (b) $E = \{x \in \mathbf{R}^2 \mid x_1 > -1, x_2 > -2, x_1 \neq 1\} \sim \{\mathbf{0}\}$.

3. $x(t) = \begin{Bmatrix} t^2/4, t \geq 0 \\ -t^2/4, t \leq 0 \end{Bmatrix}$, $x(t) = \begin{Bmatrix} 0, t \geq 0 \\ -t^2/4, t \leq 0 \end{Bmatrix}$, $x(t) = \begin{Bmatrix} t^2/4, t \geq 0 \\ 0, t \leq 0 \end{Bmatrix}$, $x(t) = 0$.

4. $x(t) = 2/\sqrt{1-8t}$ for $-\infty < t < 1/8$ and $x(t) \to \infty$ as $t \to 1/8^-$.

5. $x(t) = \sqrt{t}$ is a solution on $(0, \infty)$ but not on $[0, \infty)$ since $x'(t) = 1/2\sqrt{t}$ is undefined at $t = 0$.

6. $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| = \max\limits_{|a|=1} \sqrt{[(x_1 - y_1)a_1 + (x_2 - y_2)a_2]^2 + [(y_2 - x_2)a_1 + (x_1 - y_1)a_2]^2}$.

Thus, if $|\mathbf{x} - \mathbf{y}| < \delta$, then $|x_1 - y_1| < \delta$, $|x_2 - y_2| < \delta$ and therefore

$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| < \max\limits_{|a|=1} \delta\sqrt{(a_1 + a_2)^2 + (a_1 + a_2)^2} \leq 2\delta = \varepsilon$ if $\delta = \varepsilon/2$.

PROBLEM SET 2.2

1. (a) $u_1(t) = 1 + t$, $u_2(t) = 1 + t + t^2 + t^3/3$, $u_3(t) = 1 + t + t^2 + t^3 + 2t^4/3 + t^5/3 + t^6/9 + t^7/63$.

Mathematical induction: $u_1(t) = 1 + t$, $u_2(t) = 1 + t + t^2 + 0(t^3)$ and for $n \geq 1$, assuming $u_n(t) = 1 + t + t^2 + \cdots + t^n + 0(t^{n+1})$ we find that $u_{n+1}(t) = 1 + \int_0^t [1 + s + s^2 + \cdots + s^n +$

$0(s^{n+1})]^2 ds = 1 + \int_0^t [1 + 2s + 3s^2 + \cdots + (n+1)s^n + 0(s^{n+1})]ds = 1 + t + t^2 + \cdots + t^{n+1} + 0(t^{n+2})$. QED.

(b) By separating variables and integrating we find that $x(t) = 1/(c - t)$ and the initial condition implies that $c = 1$. For $x(t) = (1 - t)^{-1}$, we have that $\dot{x}(t) = (1 - t)^{-2} = x^2(t)$ for $t \neq 0$; and since $x(0) = 1$, $t \in (-\infty, 1)$, and this function is a solution of the IVP in part (a) according to Definition 1. The Taylor series for $x(t) = 1/(1 - t) = 1 + t + \cdots + t^n + \cdots$, which agrees with the first $(n + 1)$-terms in $u_n(t)$ found in part (a).

(c) $\dot{x}(t) = (3t)^{-2/3} = 1/x^2(t)$ for all $t \neq 0$; hence the function $x(t) = (3t)^{1/3}$ is a solution of the given differential equation on the interval $(-\infty, 0)$ or on the interval $(0, \infty)$. Clearly this function satisfies $x(1/3) = 1$, $1/3 \in (0, \infty)$ and hence $x(t) = (3t)^{1/3}$ is a solution of the given IVP on the interval $(0, \infty)$ according to Definition 1.

2. $\mathbf{u}_0(t) = \mathbf{x}_0$, $\mathbf{u}_1(t) = \mathbf{x}_0 + A\mathbf{x}_0, \cdots, \mathbf{u}_k(t) = (I + A + \cdots + A^k/k!)\mathbf{x}_0$ and $\lim_{k\to\infty} \mathbf{u}_k(t) = e^{At}\mathbf{x}_0$ absolutely and uniformly on any interval $[0, t_0]$.

5. By the lemma in this section, $\mathbf{f}$ is locally Lipschitz in E. Therefore, given $\mathbf{x}_0 \in E$, there exists a $K_0 > 0$ and an $\varepsilon > 0$ such that $N_\varepsilon(\mathbf{x}_0) \subset E$ and for all $\mathbf{x}, \mathbf{y} \in N_\varepsilon(\mathbf{x}_0)$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K_0 |\mathbf{x} - \mathbf{y}|$. Next, $T \circ \mathbf{u}(t)$ is continuous at $t = 0$. Therefore for $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|t| < \delta$ then $|T \circ \mathbf{u}(t) - T \circ \mathbf{u}(0)| = |T \circ \mathbf{u}(t) - \mathbf{x}_0| < \varepsilon$. Choose $a > 0$ such that $a < \min(\delta, 1/K_0)$. Then for $I = [-a, a]$, $t \in I$ and $\mathbf{u}, \mathbf{v} \in V \equiv \{\mathbf{u} \in C(I) \mid \|\mathbf{u} - \mathbf{x}_0\| \leq \varepsilon\}$, $|T \circ \mathbf{u}(t) - T \circ \mathbf{v}(t)| = \left|\int_0^t [\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))] ds\right| \leq \int_0^t |\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))| ds \leq c\|\mathbf{u} - \mathbf{v}\|$ where $c = K_0 a < 1$. Thus, by the contraction mapping principle, there exists a unique $\mathbf{u}(t) \in V \subset C(I)$ such that $T \circ \mathbf{u}(t) = \mathbf{u}(t)$ for all $t \in I$.

6. If $\mathbf{x}(t)$ is a continuous function on I that satisfies the integral equation, then $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(t) = \frac{d}{dt}\int_0^t \mathbf{f}(\mathbf{x}(s)) ds = \mathbf{f}(\mathbf{x}(t))$ for all $t \in I$ by the fundamental theorem of calculus since $\mathbf{f}(\mathbf{x}(t)) \in C(I)$; and therefore $\mathbf{x}(t)$ is differentiable and it satisfies the initial value problem

(2) for all $t \in I$. Conversely, if $\mathbf{x}(t)$ is a solution of the initial value problem (2) for all $t \in I$, then $\mathbf{x}(t)$ is differentiable and hence continuous on I and $\mathbf{x}(t) \in E$ for all $t \in I$; therefore, $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ implies that $\mathbf{x}(t) = \int_0^t \mathbf{f}(\mathbf{x}(s))ds + \mathbf{c}$ for all $t \in I$ and clearly $\mathbf{c} = \mathbf{x}(0) = \mathbf{x}_0$. Thus, $\mathbf{x}(t)$ satisfies the integral equation for all $t \in I$.

7. $\ddot{\mathbf{x}}(t) = \frac{d}{dt}[\mathbf{f}(\mathbf{x}(t))] = D\mathbf{f}[\mathbf{x}(t)]\dot{\mathbf{x}}(t) = D\mathbf{f}[\mathbf{x}(t)]\mathbf{f}(\mathbf{x}(t)) \in C(I)$ by the chain rule since $\mathbf{x}(t) \in E$, $D\mathbf{f}[\mathbf{x}(t)]$ and $\mathbf{f}(\mathbf{x}(t))$ are continuous for all $t \in I$.

8. Since a continuous function on a compact set is bounded, $D\mathbf{f}$ is bounded on E. It then follows immediately from Theorem 9.19 in [R] that $\mathbf{f}$ satisfies a Lipschitz condition on E.

9. Suppose that there is a constant $K_0 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in E$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le K_0 |\mathbf{x} - \mathbf{y}|$. Then, given $\varepsilon > 0$, choose $\delta = \varepsilon/K_0 > 0$ to get that for $\mathbf{x}, \mathbf{y} \in E$ with $|\mathbf{x} - \mathbf{y}| < \delta$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le K_0 |\mathbf{x} - \mathbf{y}| < K_0\delta = \varepsilon$. Therefore, $\mathbf{f}$ is uniformly continuous on E.

10.(a) Follow the hint for $\delta < 1$; and for $\delta \ge 1$, choose $x = 1$ and $y = 1/3$ to show that $|f(x) - f(y)| = 2 > 1 = \varepsilon$.

(b) Use the result of part (a) and Problem 9 to show that $f(x) = 1/x$ does not satisfy a Lipschitz condition on (0, 1).

11. If $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, then there exists a linear transformation $D\mathbf{f}(\mathbf{x}_0)$ such that given $\varepsilon = 1$, there is a $\delta > 0$ such that for $|\mathbf{x} - \mathbf{x}_0| < \delta$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)| \le |\mathbf{x} - \mathbf{x}_0|$. Thus, for $K_0 = 1 + \|D\mathbf{f}(\mathbf{x}_0)\|$, we obtain the desired result.

PROBLEM SET 2.3

1. The initial value problem has the solution $\mathbf{u}(t, \mathbf{y}) = e^{At}\mathbf{y}$. Thus, $\Phi(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y}) = e^{At}$ which is the unique fundamental matrix satisfying $\dot{\Phi} = A\Phi$ and $\Phi(0) = I$.

2. (a) $u_1(t, \mathbf{y}) = y_1 e^{-t}$, $u_2(t, \mathbf{y}) = -y_1^2 e^{-2t} + (y_1^2 + y_2)e^{-t}$ and $u_3(t, \mathbf{y}) = (-y_1^2/3)e^{-2t} + (y_1^2/3 + y_3)e^t$.

$$\Phi(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y}) = \begin{bmatrix} e^{-t} & 0 & 0 \\ -2y_1e^{-2t} + 2y_1e^{-t} & e^{-t} & 0 \\ -\frac{2}{3}y_1e^{-2t} + \frac{2}{3}y_1e^{t} & 0 & e^{t} \end{bmatrix}, \ \Phi(0) = I \text{ and}$$

$$\dot{\Phi}(t) = \begin{bmatrix} -e^{-t} & 0 & 0 \\ 4y_1e^{-2t} - 2y_1e^{-t} & -e^{-t} & 0 \\ \frac{4}{3}y_1e^{-2t} + \frac{2}{3}y_1e^{t} & 0 & e^{t} \end{bmatrix} = Df[\mathbf{u}(t, \mathbf{y})]\Phi(t) = \begin{bmatrix} -1 & 0 & 0 \\ 2y_1e^{-t} & -1 & 0 \\ 2y_1e^{-t} & 0 & 1 \end{bmatrix} \Phi(t).$$

(b) $\Phi(t) = \begin{bmatrix} (1 - y_1t)^{-2} & 0 \\ (1 - e^{t})/y_1^2 & e^{t} \end{bmatrix}$ etc. (See p. 124 in the Appendix.)

5. By the corollary in this section, it follows from Liouville's Theorem that $\det \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{x}_0) = \exp\int_0^t \text{trace } Df\,[\mathbf{u}(s, \mathbf{x}_0)]\,ds = \exp\int_0^t \nabla\cdot \mathbf{f}(\mathbf{u}(s, \mathbf{x}_0))ds$ since trace $Df = \nabla\cdot\mathbf{f}$.

6. From vector calculus, i.e., from the hint, it follows that $\mathbf{y} = \mathbf{u}(t, \mathbf{y}_0)$ is volume preserving iff $J(\mathbf{x}) = \det \frac{\partial \mathbf{u}}{\partial \mathbf{x}}(t, \mathbf{x}) = 1$ for all $t \in [0, a]$. But, from Problem 5, this follows iff $\int_0^t \nabla\cdot\mathbf{f}(\mathbf{u}(s, \mathbf{y}_0))ds = 0$ for all $t \in [0, a]$ and $\mathbf{y}_0 \in E$; and by continuity, this follows iff $\nabla\cdot\mathbf{f}(\mathbf{x}) = 0$ for all $\mathbf{x} \in E$.

PROBLEM SET 2.4

1. (a) $x(t) = x_0/(1 - x_0t)$; $(\alpha, \beta) = (-\infty, 1/x_0)$ for $x_0 > 0$ and $x(t) \to \infty$ as $t \to (1/x_0)^-$; $(\alpha, \beta) = (-\infty, \infty)$ for $x_0 = 0$; and $(\alpha, \beta) = (1/x_0, \infty)$ for $x_0 < 0$ and $x(t) \to -\infty$ as $t \to (1/x_0)^+$.

   (b) $(\alpha, \beta) = (-1, 1)$ and $x(t) = \sin^{-1}(t) \to \mp\pi/2 \in \dot{E}$ as $t \to \alpha^+$ or as $t \to \beta^-$ where $E = (-\pi/2, \pi/2)$.

(c) $x(t) = -2\tanh(2t)$ and $(\alpha, \beta) = (-\infty, \infty)$.

(d) $x(t) = |x_0|/(1 - 2x_0^2 t)^{1/2}$, $(\alpha, \beta) = (-\infty,\ 1/2x_0^2)$ and $x(t) \to \infty$ as $t \to (1/2x_0^2)^-$.

(e) $x_1(t) = y_1/(1 - y_1 t)$, $x_2(t) = (y_2 - 1 + 1/y_1)e^t + (1 + t - 1/y_1)$, $(\alpha, \beta) = (-\infty, 1/y_1)$ and $|\mathbf{x}(t)| \to \infty$ as $t \to (1/y_1)^-$.

2. (a) $x_1(t) = (1 - t)^{-1}$, $x_2(t) = t + e^t$, $(\alpha, \beta) = (-\infty, 1)$ and $|\mathbf{x}(t)| \to \infty$ as $t \to 1^-$.

(b) $x_1(t) = \sqrt{1+t}$, $x_2(t) = (1 - t)^{-1}$, $(\alpha, \beta) = (-1, 1)$, $\mathbf{x}(t) \to (0, .5)^T \in \dot{E}$ as $t \to (-1)^+$, where $E = \{x_1 > 0\}$, and $|\mathbf{x}(t)| \to \infty$ as $t \to 1^-$.

(c) $x_1(t) = \sqrt{1+t}$, $x_2(t) = (2/3)(1 + t)^{3/2} + 1/3$, $(\alpha, \beta) = (-1, \infty)$ and $\mathbf{x}(t) \to (0, 1/3)^T \in \dot{E}$ as $t \to (-1)^+$, where $E = \{x_1 > 0\}$.

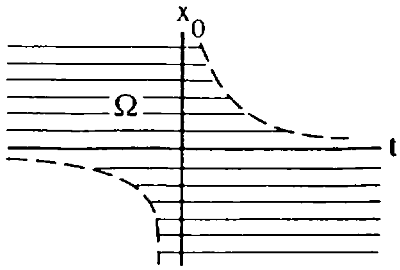
3. Asssume $\beta < \infty$. If $\lim\limits_{t\to\infty^-} \mathbf{x}(t)$ does not exist, then there exists a sequence $t_n \to \beta^-$ such that $\{\mathbf{x}(t_n)\}$ is not Cauchy; i.e., there exists an $\varepsilon > 0$ such that for all integers N, there exist integers $n > m \geq N$ such that $|\mathbf{x}(t_n) - \mathbf{x}(t_m)| \geq \varepsilon$. Thus, for $N = 1$, there exists integers $n_1 > m_1 \geq 1$ such that $|\mathbf{x}(t_{n_1}) - \mathbf{x}(t_{m_1})| \geq \varepsilon$; for $N = n_1$, there exist integers $n_2 > m_2 \geq n_1$ such that $|\mathbf{x}(t_{n_2}) - \mathbf{x}(t_{m_2})| \geq \varepsilon$; $\cdots$ for $N = n_j$, there exist integers $n_{j+1} > m_{j+1} \geq n_j$ such that $|\mathbf{x}(t_{n_j}) - \mathbf{x}(t_{m_j})| \geq \varepsilon$. Hence, the arc length of $\Gamma_+ \geq \sum\limits_{n=1}^{\infty} |\mathbf{x}(t_{n+1}) - \mathbf{x}(t_n)| \geq \sum\limits_{j=1}^{\infty} |\mathbf{x}(t_{n_j}) - \mathbf{x}(t_{m_j})| \geq \sum\limits_{j=1}^{\infty} \varepsilon = \infty$. Hence if $\beta < \infty$ and the arc length of Γ_+ is finite, it follows that $\lim\limits_{t\to\beta^-} \mathbf{x}(t)$ exists.

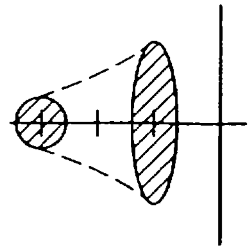
4. In cylindrical coordinates $\dot{r} = 0$, $\dot{\theta} = r^2/x_3^2 = 1/x_3^2$ and $\dot{x}_3 = 1$. Thus, $r = 1$, $x_3(t) = t + 1/\pi$ and $\theta(t) = -(t + 1/\pi)^{-1}$. $(\alpha, \beta) = (-1/\pi, \infty)$, and $\lim \mathbf{x}(t)$ as $t \to (-1/\pi)^+$ does not exist (Γ spirals down toward the unit circle in the x_1, x_2 plane as $t \to (-1/\pi)^+$); also, Γ_+ and Γ_- both have infinite arc length (cf. Problem 3).

5. Suppose $\lim\limits_{t\to\beta^-} \mathbf{x}(t) = \mathbf{x}_1 \in E$. Then since E is open, there is an $\varepsilon > 0$ such that $N_{2\varepsilon}(\mathbf{x}_1) \subset E$ and $\overline{N_\varepsilon(\mathbf{x}_1)} \subset E$. Assume that $\beta < \infty$. Then there is a $\delta > 0$ such that for $|t - \beta| < \delta$,

$|\mathbf{x}(t) - \mathbf{x}_1| < \varepsilon$. Since $\mathbf{x}(t)$ is continuous and $[0, \beta - \delta]$ is a compact set, $K = \{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{y} = \mathbf{x}(t), t \in [0, \beta - \delta]\} \cup \{\mathbf{y} \in \mathbf{R}^n \mid |\mathbf{y} - \mathbf{x}_1| \le \varepsilon\}$ is a compact subset of E; furthermore, $\Gamma_+ \subset K$. Thus, by Corollary 2, β is not finite; i.e., $\beta = \infty$. Next, we show that $\mathbf{f}(\mathbf{x}_1) = \mathbf{0}$. Suppose that $\mathbf{f}(\mathbf{x}_1) \ne 0$, say $|\mathbf{f}(\mathbf{x}_1)| = \delta > 0$. Then by the continuity of $\mathbf{f}$, there exists an $\varepsilon > 0$ such that $|\mathbf{x} - \mathbf{x}_1| < \varepsilon$ implies that $\mathbf{x} \in E$ and $|\mathbf{f}(\mathbf{x})| \ge \delta/2$. Since $\mathbf{x}(t) \to \mathbf{x}_1$ and $\dot{\mathbf{x}}(t) \to \mathbf{v}_1 \equiv \mathbf{f}(\mathbf{x}_1)$ as $t \to \infty$, it follows that for this $\varepsilon > 0$, there exists a $t_0 \ge 0$ such that for all $t \ge t_0$, $|\mathbf{x}(t) - \mathbf{x}_1| < \varepsilon$ and $|\dot{\mathbf{x}}(t) - \mathbf{v}_1| < \varepsilon$, i.e., for all $t \ge t_0$, $|\dot{\mathbf{x}}(t)| = |\mathbf{f}(\mathbf{x}(t))| \ge \delta/2$ and $|\mathbf{v}_1 \cdot \dot{\mathbf{x}}(t)| = |\mathbf{v}_1|\,|\dot{\mathbf{x}}(t)| \cdot |\cos\theta_1(t)| \ge |\mathbf{v}_1|\,\delta/4$ where $\theta_1(t)$ is the angle between $\dot{\mathbf{x}}(t)$ and $\mathbf{v}_1$ and $|\cos\theta_1(t)| \ge 1/2$ for all $t \ge t_0$. Then by the mean value theorem, for all $t > t_0$, there is a $\tilde{t} \in (t_0, t)$ such that $\mathbf{v}_1 \cdot \mathbf{x}(t) - \mathbf{v}_1 \cdot \mathbf{x}(t_0) = (t - t_0)\mathbf{v}_1 \cdot \dot{\mathbf{x}}(\tilde{t})$; thus, $|\mathbf{v}_1|\,|\mathbf{x}(t) - \mathbf{x}(t_0)| \ge |\mathbf{v}_1 \cdot [\mathbf{x}(t) - \mathbf{x}(t_0)]| = |t - t_0|\,|\mathbf{v}_1 \cdot \dot{\mathbf{x}}(\tilde{t})| \ge |t - t_0|\,|\mathbf{v}_1|\,\delta/4$ and therefore $|\mathbf{x}(t) - \mathbf{x}(t_0)| \ge |t - t_0|\,\delta/4 \ge 2\varepsilon$ for $t \ge t_0 + 8\varepsilon/\delta$. Since $|\mathbf{x}(t_0)| < \varepsilon$, this implies that for $t \ge t_0 + 8\varepsilon/\delta$, $|\mathbf{x}(t) - \mathbf{x}_1| \ge |\mathbf{x}(t) - \mathbf{x}(t_0)| - |\mathbf{x}(t_0) - \mathbf{x}_1| \ge 2\varepsilon - \varepsilon = \varepsilon$, a contradiction since $|\mathbf{x}(t) - \mathbf{x}_1| < \varepsilon$ for all $t \ge t_0$. Thus, $\mathbf{f}(\mathbf{x}_1) = \mathbf{0}$; and $\mathbf{x}_1$ is an equilibrium point of (1), i.e., $\mathbf{x}(t) = \mathbf{x}_1$ is the solution of (1) satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_1$.

PROBLEM SET 2.5





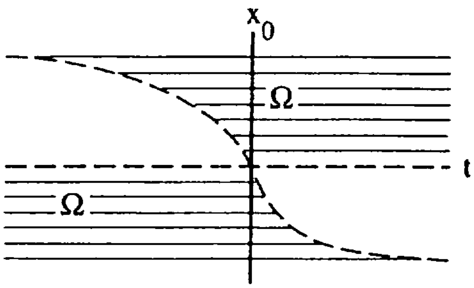
1. $x(t) = x_0/(1 - x_0t)$.

3. $\phi_t(\mathbf{x}) = \text{diag}[e^{-t}, e^{2t}]\mathbf{x}$.

5. $\phi_t(y) = (y_1e^{-t}, -y_1^2e^{-2t}/4 + (4y_2 + y_1^2)e^{2t}/4)^T$. If $y_2 = -y_1^2/4$, then $\phi_t(y) = (y_1e^{-t}, -y_1^2e^{-2t}/4) \in S$.

6. $\phi_t(y) = (y_1 e^{-t}, -y_1^2 e^{-2t} + y_2 e^{-t} + y_1^2 e^{-t}, -y_1^2 e^{-2t}/3 + (3y_3 + y_1^2)e^{t}/3).^T$ If $y_3 = -y_1^2/3$, then $\phi_t(y) = (y_1 e^{-t}, -y_1^2e^{-2t} + y_2e^{-t} + y_1^2 e^{-t}, -y_1^2e^{-2t}/3) \in S$.

7. $\phi_t(x_0) = (3t + x_0^3)^{1/3}$. For $x_0 > 0$, $(\alpha, \beta) = (-x_0^3/3, \infty)$ and $\phi_t(x_0) \to 0 \in \dot{E}$ as $t \to (-x_0^3/3)^+$.



PROBLEM SET 2.6

1. (a) (0, 0) a source, (1, 1) and (−1, 1) saddles.

   (b) (4, 2) a source, (−2, −1) a sink.

   (c) (0, 0) a source, (0, −2), ($\pm\sqrt{3}$, 1) saddles.

   (d) (0, 0, 0) a saddle.

   (e) See the hint concerning the origin. For $k > 1$, $\left(\pm\sqrt{k-1}, \pm\sqrt{k-1}, k-1\right)$ are sinks.

2. See Problem 1(e) regarding the nature of the equilibrium points of the Lorenz system; two new equilibrium points bifurcate from the equilibrium point $\mathbf{x} = \mathbf{0}$ at the bifurcation value $\mu = 1$ in a "pitchfork bifurcation." Also, see Example 5 in Section 4.5.

3. $H^{-1}(\mathbf{x}) = (x_1, x_2 - x_1^2, x_3 - x_1^2/3)^T$ is continuous on $\mathbf{R}^3$ and if $\mathbf{y} = H(\mathbf{x})$, then $\dot{\mathbf{y}} = (\dot{x}_1, \dot{x}_2 + 2x_1\dot{x}_1, \dot{x}_3 + 2x_1\dot{x}_1/3)^T = (-x_1, -x_2 - x_1^2, x_3 + x_1^2/3)^T = (-y_1, -y_2, y_3)^T = Df(0)\mathbf{y}$.

PROBLEM SET 2.7

1. $\lambda_1 = -3$, $\lambda_2 = 7$, $\mathbf{v}_1 = (3, -2)^T$, $\mathbf{v}_2 = (1, 1)^T$, $P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{y} = P^{-1}\mathbf{x}$ and $\dot{\mathbf{y}} = \text{diag}\,[-3, 7]\mathbf{y} + (-6y_1^2 + y_1y_2 + y_2^2, -9y_1^2 - 6y_1y_2 - y_2^2)^T$.

2. $\mathbf{u}^{(1)}(t, \mathbf{a}) = (e^{-t}a_1, 0)^T$, $\mathbf{u}^{(2)}(t, \mathbf{a}) = \mathbf{u}^{(3)}(t, \mathbf{a}) = (e^{-t}a_1, -e^{-2t}a_1^2/3)^T$, and $\mathbf{u}^{(j)}(t, \mathbf{a}) \to \mathbf{u}(t, \mathbf{a}) = (e^{-t}a_1, -e^{-2t}a_1^2/3)^T$. Thus, $S : x_2 = -x_1^2/3$ and $U : x_1 = 0$.

3. $\boldsymbol{\phi}_t(\mathbf{c}) = (c_1e^{-t}, -c_1^2e^{-2t}/3 + (c_1^2/3 + c_2)e^t)^T$, $S : c_2 = -c_1^2/3$, for $\mathbf{x} \in S$, $\boldsymbol{\phi}_t(\mathbf{x}) = (x_1e^{-t}, -x_1^2e^{-2t}/3)^T \in S$, and $U : x_1 = 0$.

4. $\mathbf{u}^{(1)}(t, \mathbf{a}) = (e^{-t}a_1, e^{-t}a_2, 0)^T$, $\mathbf{u}^{(2)}(t, \mathbf{a}) = (e^{-t}a_1, e^{-t}(a_2 + a_1^2) - e^{-2t}a_1^2, -e^{-2t}a_2^2/3)^T$, $\mathbf{u}^{(3)}(t, \mathbf{a}) = \mathbf{u}^{(4)}(t, \mathbf{a}) = (e^{-t}a_1, e^{-t}(a_2 + a_1^2) - e^{-2t}a_1^2, -e^{-4t}a_1^4/5 + e^{-3t}a_1^2(a_2 + a_1^2)/2 - e^{-2t}(a_2 + a_1^2)^2/3)^T$. $S : x = \psi_3(x_1, x_2)$ where $\psi_3(a_1, a_2) = u_3(0, a_1, a_2, 0) = -a_2^2/3 - a_1^2a_2/6 - a_1^4/30$; i.e., $S : x_3 = -x_2^2/3 - x_1^2x_2/6 - x_1^4/30$. To find U, let $t \to -t$ to get $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2 - x_1^2$ and $\dot{x}_3 = -x_3 - x_2^2$. For this system $\mathbf{u}^{(1)}(t, \mathbf{a}) = \mathbf{u}^{(2)}(t, \mathbf{a}) = (e^{-t}a_1, 0, 0)^T$. Thus, $U : x_1 = 0$, $x_2 = 0$, i.e., U is the x_3-axis.

5. $x_1(t) = c_1e^{-t}$, $x_2(t) = -c_1^2e^{-2t} + (c_2 + c_1^2)e^{-t}$, $x_3(t) = -c_1^4e^{-4t}/5 + c_1^2(c_2 + c_1^2)e^{-3t}/2 - (c_2 + c_1^2)^2e^{-2t}/3 + (30c_3 + c_1^4 + 5c_1^2c_2 + 10c_2^2)e^t/30$; $\lim_{t\to\infty} \boldsymbol{\phi}_t(\mathbf{c}) = \mathbf{0}$ iff $30c_3 + c_1^4 + 5c_1^2c_2 + 10c_2^2 = 0$; therefore, $S : x_3 = -x_2^2/3 - x_1^2x_2/6 - x_1^4/30$; and $\lim_{t\to-\infty} \boldsymbol{\phi}_t(\mathbf{c}) = \mathbf{0}$ iff $c_1 = c_2 = 0$; therefore $U : x_1 = 0$ and $x_2 = 0$.

6. Since $\mathbf{F} \in C^1(E)$, it follows that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\boldsymbol{\xi} \in N_\delta(\mathbf{0})$, $\|D\mathbf{F}(\boldsymbol{\xi}) - D\mathbf{F}(\mathbf{0})\| = \|D\mathbf{F}(\boldsymbol{\xi})\| < \varepsilon$. Thus, for all $\mathbf{x}, \mathbf{y} \in N_\delta(\mathbf{0})$, $|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})| \le \|D\mathbf{F}(\boldsymbol{\xi})\|\,|\mathbf{x} - \mathbf{y}| < \varepsilon\,|\mathbf{x} - \mathbf{y}|$.

7. $U_1 = \{x \in S^1 \mid y > 0\}$, $h_1(x, y) = -x$, $h_1^{-1}(x) = \left(-x, \sqrt{1-x^2}\right)^T$; $U_2 = \{x \in S^1 \mid y < 0\}$, $h_2(x, y) = x$, $h_2^{-1}(x) = \left(x, -\sqrt{1-x^2}\right)^T$; $U_3 = \{x \in S^1 \mid x > 0\}$, $h_3(x, y) = y$, $h_3^{-1}(y) = \left(\sqrt{1-y^2}, y\right)^T$; and $U_4 = \{x \in S^1 \mid x < 0\}$, $h_4(x, y) = -y$, $h_4^{-1}(y) = \left(-\sqrt{1-y^2}, y\right)^T$.

$U_1 \cap U_2 = \emptyset$, $U_3 \cap U_4 = \emptyset$; $h_3(U_1 \cap U_3) = \{y \in \mathbf{R} \mid 0 < y < 1\}$, $h_1 \circ h_3^{-1}(y) = -\sqrt{1-y^2}$ and $Dh_1 \circ h_3^{-1}(y) = y/\sqrt{1-y^2} > 0$ for $y \in h_3(U_1 \cap U_3)$; $h_4(U_1 \cap U_4) = \{y \in \mathbf{R} \mid -1 < y < 0\}$, $h_1 \circ h_4^{-1}(y) = \sqrt{1-y^2}$ and $Dh_1 \circ h_4^{-1}(y) = -y/\sqrt{1-y^2} > 0$ for $y \in h_4(U_1 \cap U_4)$; and it is similarly shown that $Dh_i \circ h_j^{-1}(x) > 0$ for $x \in h_j(U_i \cap U_j)$ for $i = 2$, $j = 3, 4$ etc.

8. $h_1 \circ h_3^{-1}(z, x) = (x, \sqrt{1-x^2-z^2})$, $h_1 \circ h_4^{-1}(x, z) = (x, -\sqrt{1-x^2-z^2})$, $h_1 \circ h_5^{-1}(y, z) = (\sqrt{1-y^2-z^2}, y)$, $h_1 \circ h_6^{-1}(z, y) = (-\sqrt{1-y^2-z^2}, y)$; $Dh_1 \circ h_3^{-1}(z, x) =$

$$\begin{bmatrix} 0 & 1 \\ \dfrac{-z}{\sqrt{1-x^2-z^2}} & \dfrac{-x}{\sqrt{1-x^2-z^2}} \end{bmatrix}, \quad Dh_1 \circ h_4^{-1}(x, z) = \begin{bmatrix} 1 & 0 \\ \dfrac{x}{\sqrt{1-x^2-z^2}} & \dfrac{z}{\sqrt{1-x^2-z^2}} \end{bmatrix},$$

$$Dh_1 \circ h_5^{-1}(y, z) = \begin{bmatrix} \dfrac{-y}{\sqrt{1-y^2-z^2}} & \dfrac{-z}{\sqrt{1-y^2-z^2}} \\ 1 & 0 \end{bmatrix},$$

$$Dh_1 \circ h_6^{-1}(z, y) = \begin{bmatrix} \dfrac{z}{\sqrt{1-y^2-z^2}} & \dfrac{y}{\sqrt{1-y^2-z^2}} \\ 0 & 1 \end{bmatrix};$$

$\det Dh_1 \circ h_3^{-1}(z, x) = \dfrac{z}{\sqrt{1-x^2-z^2}} > 0$ for $(z, x) \in h_3(U_1 \cap U_3) = \{(z, x) \in \mathbf{R}^2 \mid x^2 + z^2 < 1, z > 0\}$, $\det Dh_1 \circ h_4^{-1}(x, z) = \dfrac{z}{\sqrt{1-x^2-z^2}} > 0$ for $(x, z) \in h_4(U_1 \cap U_4) = \{(x, z) \in \mathbf{R}^2 \mid x^2 + z^2 < 1, z > 0\}$, $\det Dh_1 \circ h_5^{-1}(y, z) = \dfrac{z}{\sqrt{1-y^2-z^2}} > 0$ for $(y, z) \in h_5(U_1 \cap U_5) = \{(y, z) \in \mathbf{R}^2 \mid y^2 + z^2 < 1, z > 0\}$, $\det Dh_1 \circ h_6^{-1}(z, y) = \dfrac{z}{\sqrt{1-y^2-z^2}} > 0$ for $(z, y) \in h_6(U_1 \cap U_6) = \{(z, y) \in \mathbf{R}^2 \mid y^2 + z^2 < 1, z > 0\}$, and so forth.

Problem Set 2.8

1. Let $y_j(0) = y_{j0}$; then $y_1(t) = y_{10}e^{-t}$, $y_2(t) = y_{20}e^{-t} + z_0^2(e^{2t} - e^{-t})/3$, $z(t) = z_0 e^t$; $\Phi_0(\mathbf{y}, z) = (y_1, y_2)^T$, $\Phi_1(\mathbf{y}, z) = (y_1, y_2 - e^{-2}k_0 z^2)^T$, $\Phi_2(\mathbf{y}, z) = (y_1, y_2 - e^{-2}k_0(1 + e^{-3})z^2)^T$, $\Phi_3(\mathbf{y}, z) = (y_1, y_2 - e^{-2}k_0(1 + e^{-3} + e^{-6})z^2)^T, \cdots$, where $k_0 = (e^3 - 1)/3e$; $\Phi_k(\mathbf{y}, z) \to (y_1, y_2 - z^2/3)^T$; $\boldsymbol{\Psi}_k(\mathbf{y}, z) = z$; $H_0(\mathbf{y}, z) = (y_1, y_2 - z^2/3, z)^T$, $L^{-t}H_0T^t(\mathbf{y}, z) = (y_1, y_2 - z^2/3, z)^T$, $H(\mathbf{y}, z) = (y_1, y_2 - z^2/3, z)^T$, and $H^{-1}(\mathbf{y}, z) = (y_1, y_2 + z^2/3, z)^T$; $E^s = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = 0\}$ and $H^{-1}(E^s) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = 0\}$; $E^u = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = x_2 = 0\}$, and $H^{-1}(E^u) = \{\mathbf{x} \in \mathbf{R}^3 \mid \mathbf{x} = (0, z^2/3, z)\} = W^u(\mathbf{0})$.

2. $y(t) = y_0 e^{-t}$, $z_1(t) = z_{10}e^t$, $z_2(t) = z_{20}e^t + y_0^2(e^t - e^{-2t})/3 + y_0 z_{10}(e^t - 1)$; $\Psi_0(y, \mathbf{z}) = (z_1, z_2)^T$, $\Psi_1(y, \mathbf{z}) = (z_1, z_2 + k_0 y^2/e + k_1 y z_1/e)^T$, $\Psi_2(y, \mathbf{z}) = (z_1, z_2 + k_0 y^2(1 + e^{-3})/e + k_1 y z_1(1 + e^{-1})/e)^T$, $\Psi_3(y, \mathbf{z}) = (z_1, z_2 + k_0 y^2(1 + e^{-3} + e^{-6})/e + k_1 y z_1(1 + e^{-1} + e^{-2})/e)^T, \cdots$, where $k_0 = (e^3 - 1)/3e^2$ and $k_1 = e - 1$; $\Psi_k(y, \mathbf{z}) \to (z_1, z_2 + y^2/3 + yz_1)^T$; $\Phi_k(y, \mathbf{z}) = y$; $H(y, \mathbf{z}) = (y, z_1, z_2 + y^2/3 + yz_1)^T$, $H^{-1}(y, \mathbf{z}) = (y, z_1, z_2 - y^2/3 - yz_1)^T$; $E^s = \{\mathbf{x} \in \mathbf{R}^3 \mid x_2 = x_3 = 0\}$, $H^{-1}(E^s) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_2 = 0, x_3 = -x_1^2/3\}$; $E^u = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = 0\}$, $H^{-1}(E^u) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = 0\}$.

3. $y_1(t) = y_{10}e^{-t}$, $y_2(t) = y_{20}e^{-t} + y_{10}^2(e^{-t} - e^{-2t})$, $z(t) = z_0 e^t + y_{10}^2(e^t - e^{-2t})/3$; $\Psi_0(\mathbf{y}, z) = z$, $\Psi_1(\mathbf{y}, z) = z + k_0 y_1^2/e$, $\Psi_2(\mathbf{y}, z) = z + k_0 y_1^2(1 + e^{-3})/e$, $\Psi_3(\mathbf{y}, z) = z + k_0 y_1^2(1 + e^{-3} + e^{-6})/e, \cdots$, where $k_0 = (e^3 - 1)/3e^2$, and $\Psi_k(\mathbf{y}, z) \to z + y_1^2/3$; $\Phi_0(\mathbf{y}, z) = (y_1, y_2)^T$, $\Phi_1(\mathbf{y}, z) = (y_1, y_2 + k_1 e y_1^2)^T$, $\Phi_2(\mathbf{y}, z) = (y_1, y_2 + k_1 e y_1^2(1 + e^{-1}))^T$, $\Phi_3(\mathbf{y}, z) = (y_1, y_2 + k_1 e y_1^2(1 + e^{-1} + e^{-2}))^T, \cdots$, where $k_1 = (e - 1)/e^2$ and $\Phi_k(\mathbf{y}, z) \to (y_1, y_2 + y_1^2)^T$; $H(\mathbf{y}, z) = (y_1, y_2 + y_1^2, z + y_1^2/3)$, $H^{-1}(\mathbf{y}, z) = (y_1, y_2 - y_1^2, z - y_1^2/3)$; $E^s = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = 0\}$, $H^{-1}(E^s) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_3 = -x_1^2/3\}$; $E^u = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = x_2 = 0\}$, $H^{-1}(E^u) = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 = x_2 = 0\}$.

5. $\Psi_m(\mathbf{z}) = (z_1, z_2 + mz_1^2)^T \to (z_1, \infty)$. If $H(\mathbf{z})$ satisfies (9), then $H(e^2 z_1, e^4 z_2 + e^4 z_1^2) = \mathrm{diag}[e^2, e^4]H(\mathbf{z})$; therefore, $H_2(e^2 z_1, e^4 z_2 + e^4 z_1^2) = e^4 H_2(z_1, z_2)$ and $e^4\, \partial H_2/\partial z_1(z_1, z_2) = \partial H_2/\partial z_1(e^2 z_1, e^4 z_2 + e^4 z_1^2)\cdot e^2 + \partial H_2/\partial z_2(e^2 z_1, e^4 z_2 + e^4 z_1^2)\cdot 2e^4 z_1$; setting $z_1 = z_2 = 0$
If there exists a C^1 function H satisfying (9), then
implies that $\partial H_2/\partial z_1(0, 0) = 0$. a second differentiation with respect to z_1 yields $e^4\, \partial^2 H_2/\partial z_1^2(0, 0) = e^2[\partial^2 H_2/\partial z_1^2\cdot e^2 + \partial^2 H_2/\partial z_1 \partial z_2 \cdot 2e^4 z_1] + 2e^4 z_1\, [\partial^2 H_2/\partial z_2 \partial z_1 \cdot e^2 + \partial^2 H_2/\partial z_2^2 \cdot 2e^4 z_1] + 2e^4\, \partial H_2/\partial z_2$, the right-hand side being evaluated at $(e^2 z_1, e^4 z_2 + e^4 z_1^2)$; setting $z_1 = z_2 = 0$ then implies that $\partial^2 H_2/\partial z_1^2(0, 0) = \partial^2 H_2/\partial z_1^2(0, 0) + 2\partial H_2/\partial z_2(0, 0)$, i.e., that $\partial H_2/\partial z_2(0, 0) = 0$. Thus $J(\mathbf{z}) = \det DH(\mathbf{z}) = 0$ at $\mathbf{z} = \mathbf{0}$. Finally, if H^{-1} exists, then $H \circ H^{-1}(\mathbf{z}) = \mathbf{z}$ and then by the chain rule, if H^{-1} were differentiable at $\mathbf{z} = \mathbf{0}$ we would get $DH(H^{-1}(\mathbf{z}))\cdot DH^{-1}(\mathbf{z}) = I$ which would imply that $0 = \det DH(\mathbf{0})\cdot DH^{-1}(\mathbf{0}) = 1$, a contradiction. [This contradicts Hartman's Theorem, p. 123. Therefore, there does not exist a C^2 function H satisfying (9).]

PROBLEM SET 2.9

1. (a, c, d) all unstable, (b) (4, 2) is unstable and (–2, –1) is asymptotically stable, (e) $\mathbf{0}$ is asymptotically stable for $k \le 1$ and for $k > 1$, $\mathbf{0}$ is unstable and $(\pm\sqrt{k-1}, \pm\sqrt{k-1}, k-1)$ are asymptotically stable.

2. (a) (1, 0) is an unstable proper node and (–1, 0) is an unstable saddle.

   (b) (–1, –1) and (2, 2) are unstable saddles, $(\sqrt{2}, 0)$ is an asymptotically stable proper node and $(-\sqrt{2}, 0)$ is an unstable proper node.

   (c) (1, 0) is an unstable saddle and (0, 2) is an asymptotically stable node.

4. (a) $\dot{V}(\mathbf{x}) < 0$ for $\mathbf{x} \ne \mathbf{0}$ so $\mathbf{0}$ is asymptotically stable.

   (b) $\dot{V}(\mathbf{x}) > 0$ for $\mathbf{x} \ne \mathbf{0}$ so $\mathbf{0}$ is unstable.

   (c) $\dot{V}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^2$ so $\mathbf{0}$ is a stable equilibrium point which is not asymptotically stable and solution curves lie on circles centered at the origin.

5. (a) For $V(\mathbf{x}) = x_1^2 + x_2^2$, $\dot{V}(\mathbf{x}) < 0$ for $\mathbf{x} \neq \mathbf{0}$; therefore, $\mathbf{0}$ is asymptotically stable.

(b) For $V(\mathbf{x}) = x_1^2 + x_2^2$, it follows that on any given straight line $x_2 = mx_1$ with $|m - 2| < \sqrt{3}$, $\dot{V}(\mathbf{x}) < 0$ for all sufficiently small $|\mathbf{x}| \neq 0$ and on any given straight line $x_2 = mx_1$ with $|m - 2| > \sqrt{3}$, $\dot{V}(\mathbf{x}) > 0$ for all sufficiently small $|\mathbf{x}| \neq 0$; i.e., $\mathbf{0}$ is a saddle and is unstable. This follows more easily from the Hartman-Grobman theorem since the eigenvalues of the linear part $\lambda = 1 \pm \sqrt{3}$. (Also, see p. 126 in the appendix.)

(c) For $V(\mathbf{x}) = x_1^2 + 2x_2^2/3$, it follows that $\dot{V}(\mathbf{x}) < 0$ for $0 < |\mathbf{x}| < 1$; therefore, $\mathbf{0}$ is asymptotically stable. This also follows from the Hartman-Grobman theorem since the eigenvalues of the linear part $\lambda = -2 \pm i\sqrt{5}$.

(d) For $V(\mathbf{x}) = (x_1 - x_2 - 4)^4 \cdot \exp[(x_1x_2 + x_1 - x_2 + 12) / (4 + x_2 - x_1)]$, $\dot{V}(\mathbf{x}) \equiv 0$ and therefore $\mathbf{0}$ is a center. This Liapunov function can be found by making the rotation of coordinates $u = x_1 + x_2$, $v = x_1 - x_2$ to get $du/dv = (u^2/2 + v^2/2 + 4v)/(uv - 4u)$; and then letting $w = u^2$ to get $dw/dv = (w + 8v + v^2) / (v - 4)$, a linear differential equation. The solution of this linear differential equation then yields the Liapunov function $V(x_1, x_2)$. Also, note that the u, v system is symmetric with respect to the v-axis; cf. Theorem 6 in Section 2.10.

7. Let $x_1 = x$ and $\dot{x}_2 = -g(x_1)$. Then $\ddot{x} + f(x)\,\dot{x} + g(x) = 0$ is equivalent to $\ddot{x}_1 = -g(x_1) - f(x_1)\dot{x}_1 = \dot{x}_2 - F'(x_1)\dot{x}_1$ since $F'(x_1) = f(x_1)$. And this last equation is (up to an arbitrary constant) equivalent to $\dot{x}_1 = x_2 - F(x_1)$. Let $V(\mathbf{x}) = x_2^2/2 + G(x_1)$. Then $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ if $G(x) > 0$ and $\dot{V}(\mathbf{x}) = -g(x_1)\,F(x_1) < 0$ if $g(x)\,F(x) > 0$. And since $g(0) = 0$, we have $\dot{V}(\mathbf{x}) \leq 0$ with $\dot{V}(\mathbf{x}) = 0$ on the x_2-axis. Thus, Theorem 3 implies that $\mathbf{0}$ is stable. To show that $\mathbf{0}$ is asymptotically stable, we may apply LaSalle's Invariance Principle: Let K be a bounded and positively invariant region in $\mathbf{R}^n$ and suppose that $V(\mathbf{x})$ is defined on K and that $\dot{V}(\mathbf{x}) \leq 0$ in K. Let L be the subset of K where $\dot{V}(\mathbf{x}) = 0$ and let M be the largest invariant subset of L. Then the ω-limit set of every orbit starting in K is in M and the orbit approaches M as $t \to \infty$. Cf. [67], p. 30. In this problem, we can let $K = N(\mathbf{0})$, $L = K \cap \{x_1 = 0\}$ and then $M = \{\mathbf{0}\}$ since orbits are transverse to the x_2-axis for $x_2 \neq 0$. Thus, by the above principle, $\mathbf{0}$ is asymptotically stable.

8. $F(x) = \varepsilon(x^3 - 3x)/3$, $G(x) = x^2/2 > 0$ for $x \neq 0$, and $g(x)\,F(x) = \varepsilon x^2(x^2 - 3)/3 < 0$ for $\varepsilon > 0$ and $0 < |x| < \sqrt{3}$; therefore, for $\varepsilon > 0$ the origin is an unstable equilibrium point of the van der Pol equation (using LaSalle's Invariance Principle).

PROBLEM SET 2.10

1. (a) $\dot{r} = r$, $\dot{\theta} = 1$; the origin is an unstable focus.

   (b) $\dot{r} = ry^2$, $\dot{\theta} = 1$; the origin is an unstable focus.

   (c) $\dot{r} = (x^6 + y^6)/r > 0$ and $\dot{\theta} = 1 + xy(y^4 - x^4) / r^2 > 0$ for sufficiently small $r > 0$; the origin is an unstable focus.

3. Let $F(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{0})\mathbf{x}$. Then according to the definition of differentiability, Definition 1 in Section 2.1, $|\mathbf{F}(\mathbf{x})| / |\mathbf{x}| \to 0$ as $|\mathbf{x}| \to 0$, i.e., as $\mathbf{x} \to \mathbf{0}$.

4. (a) (0, 0) is an unstable proper node, (1, 1) and (–1, 1) are topological saddles.

   (b) (4, 2) is an unstable node and (–2, –1) is a stable focus.

   (c) (0, 0) is an unstable proper node and (0, –2), $(\pm\sqrt{3}, 1)$ are topological saddles.

   (d) (0, 1) is a center since the system is symmetric with respect to the y-axis and (0, –1) is a topological saddle.

   (e) (0, ±1) are centers since the system is Hamiltonian and also since it is symmetric with respect to the y-axis and (±1, 0) are topological saddles.

   (f) (1, 0) is an unstable node and (–1, 0) is a topological saddle.

PROBLEM SET 2.11

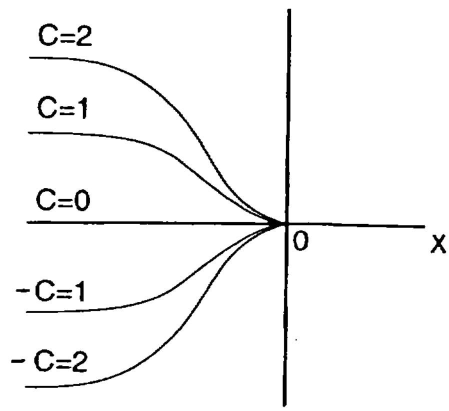
1. In Theorem 2, $n = m = 1$ is an odd integer, $b_1 = 4 \neq 0$ and $\lambda = 8 > 0$; therefore the system has a critical point with an elliptic domain at the origin. For $V(\mathbf{x}) = y - x^2/(2 \pm \sqrt{2})$ we have $\dot{V}(\mathbf{x}) \equiv 0$ on $y = x^2/(2 \pm \sqrt{2})$; thus $y = x^2/(2 \pm \sqrt{2})$ are invariant curves of the system. This system is best understood by drawing its global phase portrait; cf. Section 3.11, Prob. 5.

2. (a, b, e, f) $\mathbf{0}$ is a saddle-node. (c) $\mathbf{0}$ is a node (and it is unstable). (d) $\mathbf{0}$ is a topological saddle.

3. (a, b) **0** is a cusp. (c) **0** is a saddle-node. (d) **0** is a focus or center according to Theorem 2 and using $V(x) = x^4 + 2y^2$ with $\dot{V}(x) = -4x^2y^2$, it is a stable focus. (e) **0** is a topological saddle. (f) **0** is a focus or center according to Theorem 2; use the coordinate ransformation $\xi = x$, $\eta = x + y$ to put the system into the normal form (3). Also, it can be shown that **0** is a stable focus.

PROBLEM SET 2.12

1. Substituting $h(x) = a_2x^2 + a_3x^3 + \cdots$ into (5) yields $a_2 = 0$ and $na_n + a_{n+1} = 0$ for integer $n \geq 2$; and this implies that $a_1 = a_2 = \cdots = 0$, i.e., that $h(x) \equiv 0$. For the function h(x, c) given in this problem, we have $h'(x, c) = 0$ for $x \geq 0$ and $h'(x, c) = -ce^{1/x}/x^2$ for $x < 0$. Substitution into equation (5) yields $0 = 0$ for $x \geq 0$ and $-ce^{1/x}/x^2[x^2] - (-ce^{1/x}) = 0$ for $x < 0$; i.e., h(x, c) satisfies equation (5) for all $x \in \mathbf{R}$. Also, since h(x, c) is (real) analytic at each point $x \neq 0$ with $h^{(n)}(x, c) \to 0$ as $x \to 0$ and since $h^{(n)}(0, c) = 0$ for all $n = 1, 2, \cdots$, it follows that $h(x, c) \in C^\infty(\mathbf{R})$.



2. Diagonalization yields a system of the form $\dot{x} = \alpha(x + y)^2 - y(x + y)$, $\dot{y} = -y - \alpha(x + y)^2 + y(x + y)$; then from (5), $h(x) = -\alpha x^2 + \alpha x^3 + \cdots$ and on $W^c(\mathbf{0})$, $\dot{x} = \alpha x^2 + 0(x^3)$; so for $\alpha \neq 0$, **0** is a saddle-node. For $\alpha = 0$, $h(x) \equiv 0$ and the x-axis is a line of critical points.

3. The linear part of the system is already in diagonal form and from (5), $h(x) = -x^2 - 2x^4 + \cdots$; on $W^c(\mathbf{0})$, $\dot{x} = -x^3 + \cdots$ and the origin is a stable node.

4. From (5) we have for $h(x) = a_2x^2 + a_3x^3 + \cdots$ that $(2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots)(-x^3) + (a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots) - x^2 = 0$ identically in x for $|x| < \delta$. Therefore setting the coefficients of like powers of x equal to zero yields $a_2 = 1$, $a_3 = 0$, $2a_2 = a_4$, $a_5 = 0$, $4a_4 = a_6$, $a_7 = 0$, $\cdots$, i.e., $a_2 = 1$, $a_4 = 2a_2 = 2$, $a_6 = 4a_4 = 2.4$, $\cdots$, $a_{2n} = 2^n n!$ and $a_{2n+1} = 0$. Thus,

$h(x) = \sum_{n=0}^{\infty} 2^n n!\, x^{2n+2}$ diverges except at $x = 0$ and this polynomial system has no analytic center manifold. However, since $\dot{x} = -x^3 < 0$ for $x > 0$ and $\dot{x} = -x^3 > 0$ for $x < 0$, any trajectory $\gamma^{\pm}(t)$ with $\gamma^{\pm}(0) = (x^{\pm}(0)), y^{\pm}(0))$ and $x^{+}(0) > 0$ or $x^{-}(0) < 0$ can be represented by a function $y = f^{\pm}(x)$ which is analytic for $x > 0$ or $x < 0$ respectively. And since $W^c(\mathbf{0})$ is invariant under the flow, it follows from Theorem 1 that given $f^{+}(x)$, there exists an $f^{-}(x)$ such that the function $h(x) = \{f^{+}(x) \text{ for } x > 0, 0 \text{ for } x = 0, f^{-}(x) \text{ for } x < 0\}$ represents a C^{∞} center manifold, $W^c(\mathbf{0})$, and $\dot{x} = -x^3$ on $W^c(\mathbf{0})$; thus, the origin is a stable node.

5. (a) From (5), $h(\mathbf{x}) = -x_1^2 - x_2^2 + \cdots$; on $W^c(\mathbf{0})$, $\dot{x}_1 = -x_2 + 0(|\mathbf{x}|^3)$, $\dot{x}_2 = x_1 + 0(|\mathbf{x}|^3)$ and the origin is topologically a stable focus on $W^c(\mathbf{0})$ which follows using the Liapunov function $V(\mathbf{x}) = (x_1^2 + x_2^2)/2$ or by showing that $\dot{r} = -r^3 + 0(r^4)$ and $\dot{\theta} = 1 + 0(r)$ for the system on $W^c(\mathbf{0})$; hence, $\mathbf{0}$ is a symptotically stable critical point.

   (b) There is a saddle-node at the origin on $W^c(\mathbf{0})$.

   (c) There is a critical point with two hyperbolic sectors at the origin on $W^c(\mathbf{0})$.

6. Let $h(x) = a_2x^2 + a_3x^3 + \cdots$; then from (5), $h(x) = dx^2 + (ed - 2ad)x^3 + \cdots$ and on $W^c(0)$, $\dot{x} = ax^2 + bdx^3 + 0(x^4)$. Thus, for $a \neq 0$, the origin is a saddle-node; for $a = 0$ and $bd > 0$, the origin is a saddle; for $a = 0$ and $bd < 0$, the origin is a stable node; for $a = b = 0$ and $cd \neq 0$, $\dot{x} = cd^2x^4 + 0(x^5)$ on $W^c(\mathbf{0})$ and the origin is a saddle-node. If $a = d = 0$, the x-axis consists of critical points.

7. $h(\mathbf{x}) = x_1^2 + 0(|\mathbf{x}|^3)$; on $W^c(\mathbf{0})$, $\dot{x}_1 = -x_1^3 - x_2^3 + 0(|\mathbf{x}|^4)$; $\dot{x}_2 = x_1^3 - x_2^3 + 0(|\mathbf{x}|^4)$ and the origin is topologically a stable focus on $W^c(\mathbf{0})$ which follows using the Liapunov function $V(\mathbf{x}) = (x_1^4 + x_2^4)/4$; hence $\mathbf{0}$ is an asymptotically stable critical point.

PROBLEM SET 2.13

1. $L_J[\mathbf{h}_2(\mathbf{x})] = (b_{20}x^2 + (b_{11} - 2a_{20})xy + (b_{02} - a_{11})y^2, -2b_{20}xy - b_{11}y^2)^T$ and for $a_{02} = a_{11} = 0$, $a_{20} = (b + f)/2$, $b_{02} = -c$, $b_{11} = f$ and $b_{20} = -a$, $L_J[\mathbf{h}_2(\mathbf{x})] + \mathbf{F}_2(\mathbf{x}) = (0, dx^2 + (e + 2a)xy)^T$.

2. Since $L_J[\mathbf{h}_3(\mathbf{x})] = b_{30}(x^3, -3x^2y)^T + b_{21}(x^2y, -2xy^2)^T + b_{12}(xy^2, -y^3)^T + (b_{03} - a_{12})(y^3, 0)^T - 3a_{30}(x^2y, 0)^T - 2a_{21}(xy^2, 0)^T$, the result for $L_J(H_3)$ follows; and then it is clear that $H_3 = L_J(H_3) \oplus G_3$.

3. As in the paragraph preceding Remark 1, for $\mathbf{F}_2 = \tilde{\mathbf{F}}_2 = 0$, the system $\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{F}_3(\mathbf{x}) + O(|\mathbf{x}|^4)$ can be reduced, by letting $\mathbf{x} = \mathbf{y} + \mathbf{h}_3(\mathbf{y})$, to a system of the form $\dot{\mathbf{y}} = J\mathbf{y} + \tilde{\mathbf{F}}_3(\mathbf{y}) + O(|\mathbf{x}|^4)$ with $\tilde{\mathbf{F}}_3 \in G_3$, i.e., to a system of the form $\dot{x} = y + O(|\mathbf{x}|^4)$, $\dot{y} = ax^3 + bx^2y + O(|\mathbf{x}|^4)$ for $a, b \in \mathbf{R}$. And letting $y + O(|\mathbf{x}|^4) \to y$, we get a system of the form (3) in Section 2.11; according to Theorem 2 in 2.11, for $a > 0$ there is a topological saddle at the origin and for $a < 0$ there is a focus or a center at the origin.

4. Similar to Problem 3, we get a system of the form (3) in 2.11: $\dot{x} = y$, $\dot{y} = ax^4 + bx^3y + O(|\mathbf{x}|^5)$ which, for $a \neq 0$, has a cusp at the origin.

5. For $x_1 = y_1$ and $x_2 = y_2 - y_1^2$, the given system reduces to $\dot{x} = y - x^3 + xy^2 - y^3 + O(|\mathbf{x}|^4)$, $\dot{y} = x^2 + 3x^3 + x^2y + O(|\mathbf{x}|^4)$ and then for $\mathbf{x} = (y_1, y_2 + y_1^3 - y_1y_2^2 + y_2^3)^T$ or $\mathbf{y} = (x_1, x_2 - x_1^3 + x_1x_2^2 - x_2^3)^T$, this system reduces to $\dot{y}_1 = y_2 + O(|\mathbf{x}|^4)$; $\dot{y}_2 = y_1^2 + 3y_1^3 - 2y_1^2y_2 + O(|\mathbf{x}|^4)$.

PROBLEM SET 2.14

1. (a) $H(x, y) = a_{11}xy + a_{12}y^2/2 - a_{21}x^2/2 + Ax^2y - Bxy^2 + Cy^3/3 - Dx^3/3$. (b) If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is Hamiltonian, then $\mathbf{f} = (H_y, -H_x)$ for $\mathbf{x} \in E$ and therefore $\nabla \cdot \mathbf{f} = \partial H_y/\partial x - \partial H_x/\partial y = 0$ for $\mathbf{x} \in E$. On the other hand, if $\nabla \cdot \mathbf{f} = 0$, i.e., if $\partial f_1/\partial x = -\partial f_2/\partial y$ in a simply connected region E, then the first-order differential equation $-f_2 dx + f_1 dy = 0$ is exact. (See, for example, Theorem 2.8.1 in W.E. Boyce and R.C. Di Pima, "Elementary Differential Equations and Boundary Value Problems," J. Wiley, NY, 1997.) Thus, there exists a function $H \in C^2(E)$ such that $dH = H_x dx + H_y dy = -f_2 dx + f_1 dy$ and therefore the system $\dot{x} = f_1 = H_y$, $\dot{y} = f_2 = -H_x$ is Hamiltonian on E.

2. $H(x, y) = T(y) + U(x) = y^2/2 + x^2/2 - x^3/3$; U(x) has a strict local minimum at $x = 0$ and a strict local maximum at $x = 1$; and therefore the Hamiltonian system has a center at (0, 0) and a saddle at (1, 0).

3. $H(x, y) = y^2/2 + x^2/2 - x^4/4$; there is a center at (0, 0) and saddles at $(\pm 1, 0)$.

5. (a) The Hamiltonian system has a center and the gradient system has a stable node at (0, 0).

   (c) The Hamiltonian and gradient systems have saddles at $(n\pi, 0)$ for $n \in \mathbf{Z}$.

   (e) The Hamiltonian system has a center and the gradient system has a stable node at $(-4/3, -2/3)$.

6. (a) The surfaces $V(x, y, z) = $ constant are paraboloids with their vertices on the z-axis and trajectories, other than the z-axis, approach the positive z-axis asymptotically as $t \to \infty$.

   (b) The surfaces $V(x, y, z) = $ constant are concentric ellipsoids and the origin is a stable, three-dimensional node.

   (c) Each of the surfaces $V(x, y, z) = $ constant has a strict local maximum at the origin, a strict local minimum at (2/3, 4/3, 0) and saddles at (2/3, 0, 0) and (0, 4/3, 0); the gradient system has a source at the origin, a sink at (2/3, 4/3, 0) and saddles at (2/3, 0, 0) and (0, 4/3, 0).

7. Since $\mathbf{x}_0$ is a strict local minimum of $V(\mathbf{x})$, there is a $\delta > 0$ such that $V(\mathbf{x}) - V(\mathbf{x}_0) > 0$ for $0 < |\mathbf{x}| < \delta$ and $d/dt[V(\mathbf{x}) - V(\mathbf{x}_0)] = [\partial V/\partial \mathbf{x}]\cdot\dot{\mathbf{x}} = -[(\partial V/\partial x_1)^2 + \cdots + (\partial V/\partial x_n)^2] < 0$ for $0 < |\mathbf{x}| < \delta$.

9. First of all $(x_0, 0)$ is a critical point of the Newtonian system (3) iff $U'(x_0) = 0$. Since $\det Df(x_0, 0) = U''(x_0)$ and trace $Df(x_0, 0) = 0$, it follows that $(x_0, 0)$ is a saddle of the Newtonian system (3) if $U''(x_0) < 0$, i.e., if x_0 is a strict local maximum of $U(x)$; and since (3) is symmetric with respect to the x-axis, it follows that $(x_0, 0)$ is a center of the Newtonian system (3) if $U''(x_0) > 0$, i.e., if x_0 is a strict local minimum of $U(x)$; finally, if x_0 is a horizontal inflection point of $U(x)$, then $U'(x_0) = 0$ and the first nonvanishing derivative of $U(x)$ at x_0 is odd; therefore, it follows from Theorem 3 in Section 2.11 that $(x_0, 0)$ is a cusp for the Newtonian system (3).

11. Let $x_1 = x$, $x_2 = y$, $y_1 = \dot{x}$ and $y_2 = \dot{y}$. The two-body problem is a Hamiltonian system with $H(x_1, x_2, y_1, y_2) = (y_1^2 + y_2^2)/2 - (x_1^2 + x_2^2)^{-1/2}$. The gradient system orthogonal to this system is $\dot{x}_1 = -x_1/(x_1^2 + x_2^2)^{3/2}$, $\dot{x}_2 = -x_2/(x_1^2 + x_2^2)^{3/2}$, $\dot{y}_1 = -y_1$, $\dot{y}_2 = -y_2$.

12. By Problem 1(b), if $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is Hamiltonian, then $\nabla\cdot\mathbf{f} = 0$ in E (even if E is not simply connected) and then by Problem 6 in Section 2.3, the flow defined by this system is area preserving.

3. NONLINEAR SYSTEMS: GLOBAL THEORY

PROBLEM SET 3.1

1. $$\phi(t, \mathbf{x}) = e^{At}\mathbf{x} = \begin{bmatrix} e^{-t} & \left(e^{-t} - e^{2t}\right) \\ 0 & e^{2t} \end{bmatrix}\mathbf{x}.$$

2. The differential equation $\dot{x} = x^2/(1 + x^2)$ is separable; its solution is $x(t) = \left(t + c \pm \sqrt{(t+c)^2 + 4}\right)/2$; for $x_0 \neq 0$, $x(0) = x_0$ if $c = x_0 - 1/x_0$ and the $\pm$ sign is chosen as $x_0/|x_0|$ and this yields the result in Example 1; for $x_0 = 0$ the solution is $x(t) \equiv 0$.

3. If $f(x) \neq 0$ at $x \in E$, then $D|f(x)| = |f(x)|\, f'(x)/f(x)$; and this then yields $DF(x) = f'(x)/(1 + |f(x)|)^2$; if $f(x_0) = 0$ at $x_0 \in E$, then $DF(x_0) = \lim_{h\to 0} [F(x_0 + h) - F(x_0)]/h = \lim_{h\to 0} f(x_0 + h)/(1 + |f(x_0 + h)|)/h = f'(x_0)$ and then $\lim_{x\to x_0} DF(x) = \lim_{x\to x_0} f'(x)/(1 + |f(x)|)^2 = f'(x_0)$ since $f' \in C(E)$ and since $f(x_0) = 0$; hence $F \in C^1(E)$.

4. $f'(x) = -2x/(1 + x^2)^2$ and $f'(x)$ assumes its maximum/minimum at $x = \mp 1/\sqrt{3}$; thus $|f'(x)| \le |f'(\pm 1/\sqrt{3})| = 3\sqrt{3}$; then by the mean value theorem $|f(x) - f(y)| \le 3\sqrt{3}\, |x - y|$ for $x, y \in \mathbf{R}$. The differential equation $\dot{x} = 1/(1 + x^2)$ is separable and its solution satisfying $x(0) = x_0$ is given by the solution of the cubic $x^3 + 3x - (3t + k_0) = 0$ with $k_0 = x_0^3 + 3x_0$; the solution of this cubic is $x(t) = \left\{\left[(3t + k_0) + \sqrt{(3t+k_0)^2 + 4}\right]^{1/3} + \left[(3t + k_0) - \sqrt{(3t+k_0)^2 + 4}\right]^{1/3}\right\}/2^{1/3}$ and $x(t) \to \pm\infty$ as $t \to \pm\infty$.

6. (a) If $\mathbf{x}_0$ is an equilibrium point of (1) then $\phi_t(\mathbf{x}_0) = \mathbf{x}_0$ for all $t \in \mathbf{R}$; and since $\tau(\mathbf{x}_0, t)$ maps $\mathbf{R}$ onto $\mathbf{R}$, it follows that $\psi_\tau(H(\mathbf{x}_0)) = H(\phi_t(\mathbf{x}_0)) = H(\mathbf{x}_0)$ for all $\tau \in \mathbf{R}$; i.e., $H(\mathbf{x}_0)$ is an equilibrium point of (2). Alternatively, one may follow the hint given in Problem 6.

(b) If $\phi_t(x_0)$ is a periodic solution of (1) with period t_0, then $\phi_{t_0}(x_0) = x_0$ and therefore if $\tau_0 = \tau(x_0, t_0)$, it follows that $\psi_{\tau_0}(H(x_0)) = H(\phi_{t_0}(x_0)) = H(x_0)$, i.e., $\psi_\tau(H(x_0))$ is a periodic solution of (2) of period τ_0.

7. (Cf. [Wi], p. 25–26.) Differentiating (*) wih respect to t yields $DH(\phi_t(x))\partial\phi_t(x)/\partial t = \partial\tau(x, t)/\partial t \cdot \partial\psi_\tau(H(x))/\partial\tau$ which at $t = 0$ yields $DH(x)f(x) = \partial\tau(x, 0)/\partial t \cdot g(H(x))$. Then differentiating this last equation with respect to x yields $D^2H(x)f(x) + DH(x)Df(x) = \partial\tau(x, 0)/\partial t \cdot Dg(H(x))DH(x) + \partial^2\tau(x, 0)/\partial x\partial t \cdot g(H(x))$. And then setting $x = x_0$, this yields $ADf(x_0)A^{-1} = \partial\tau(x_0, 0)/\partial t \cdot Dg(H(x_0))$. Thus, the eigenvalues of $Df(x_0)$ and the eigenvalues of $Dg(H(x_0))$ are related by the positive constant $k_0 = \partial\tau(x_0, 0)/\partial t$.

9. For $F(x, y) = (y, \mu x + y - y^3)$ and $\mu \neq 0$, $F^{-1}(x, y) = (y - x + x^3, \mu x)/\mu$;
$DF(x, y) = \begin{bmatrix} 0 & 1 \\ \mu & 1-3y^2 \end{bmatrix}$ and $DF^{-1}(x, y) = \begin{bmatrix} \left(-1+3x^2\right)/\mu & 1/\mu \\ 1 & 0 \end{bmatrix}$ are continuous; and an easy computation yields $F\left(\sqrt{\mu}, \sqrt{\mu}\right) = \left(\sqrt{\mu}, \sqrt{\mu}\right)$.

PROBLEM SET 3.2

1. There is a saddle at (0, 0) and stable nodes at $(\pm 1, 0)$. $[-1, 1]$ is an attracting set, but it is not an attractor since it does not contain a dense orbit. $(0, 1]$ is not an attractor since it is not closed. $[1, \infty)$ is an attractor. $(0, \infty)$, $[0, \infty)$ and $(-1, \infty)$ are not attracting sets. $[-1, \infty)$ and $(-\infty, \infty)$ are attracting sets.

2. (a) By the theorem of Hurwitz given in this problem, for any irrational number α and any integer $N > 0$, there are positive integers m, n such that $n > N$ and $|\alpha n - m| < 1/n$. Furthermore, for any $\varepsilon > 0$, if we choose $N \geq 2\pi/\varepsilon$, then $|2\pi\alpha n - 2\pi m| < 2\pi/n \leq \varepsilon$ and then for $a = \exp[2\pi i\alpha]$, $|a^n - 1| < \varepsilon$. Let $\theta = 2\pi n\alpha - 2\pi m$; then $0 < |\theta| < \varepsilon$ and there exists an integer K such that $K|\theta| < 2\pi < (K + 1)|\theta|$. Thus, for any point a_0 on the unit circle C, there is an integer $j \in \{1, \cdots, K\}$ such that $|a^j - a_0| < \varepsilon$; therefore $\{a^k \mid k = 1, 2, \cdots\}$ is dense in C.

(b) The flow $\phi_t(w, z) = (e^{2\pi i t}w, e^{2\pi\alpha i t}z)$; it follows that $\phi_n(w, z) = (e^{2\pi i n}w, e^{2\pi\alpha n i t}z) = (w, a^n z)$.

(c) Let $x_0 = (w_0, z_0) \in T^2$. Given any point $(w_1, z_1) \in T^2$, let $t_0 = \arg w_1 - \arg w_0$. Then $e^{2\pi i t_0}w_0 = w_1$ and for $\tilde{z}_0 = e^{2\pi\alpha i t_0}z_0$, $\phi_{t_0}(w_0, z_0) = (w_1, \tilde{z}_0)$ since $\phi_t(w_0, z_0) = (e^{2\pi i t}w_0, e^{2\pi\alpha i t}z_0)$. Then for any $z_1 \in C$ and $\varepsilon_n = 1/n$, there is a positive integer $k_n > n$ such that $|\tilde{z}_0 \exp[2\pi\alpha i k_n] - z_1| < 1/n$; this follows from part (a) with $a_0 = z_1/\tilde{z}_0$ and $\varepsilon = 1/n$. Thus, for any point $(w_1, z_1) \in T^2$, if we let $t_n = t_0 + k_n$, then $t_n \to \infty$ and $\phi_{t_n}(w_0, z_0) = \phi_{k_n} \circ \phi_{t_0}(w_0, z_0) = \phi_{k_n}(w_1, \tilde{z}_0) \to (w_1, z_1)$ as $n \to \infty$; therefore, $(w_1, z_1) \in \omega(\Gamma_{x_0})$; i.e., $\omega(\Gamma_{x_0}) = T^2$. Similarly, it is shown that $\alpha(\Gamma_{x_0}) = T^2$.

(d) Any trajectory of this system is a solution of the Hamiltonian system with two degrees of freedom $\dot{x} = -2\pi\alpha y$, $\dot{y} = 2\pi\alpha x$; $\dot{u} = -2\pi v$, $\dot{v} = 2\pi u$; with $H(x) = -\pi[\alpha(x^2 + y^2) + (u^2 + v^2)]$. Thus, trajectories lie on the ellipsoidal surfaces $E_k = \{x \in \mathbf{R}^4 \mid \alpha(x^2 + y^2) + (u^2 + v^2) = k^2\}$. For a given $k \in \mathbf{R}$ and $x_0 \in E_k$, it follows from part (c) that $\omega(\Gamma_{x_0})$ is the torus $T^2_{h,k} = \{x \in \mathbf{R}^4 \mid u^2 + v^2 = h^2, x^2 + y^2 = (k^2 - h^2)/\alpha\} = C_h \times C_{h'}$ with $h' = \sqrt{(k^2 - h^2)/\alpha}$ and, as in Section 3.6, for a given $k \in \mathbf{R}$, we can project from the north pole of the surface E_k to obtain the projection of the tori $T^2_{h,k}$ onto $\mathbf{R}^3$; cf. Figure 5 in Section 3.6.

3. Reflexive: $\Gamma_1 \sim \Gamma_1$ since $\phi_t(x_1) = \phi_{t+t_0}(x_1)$ for $t_0 = 0$. Symmetric: If $\Gamma_1 \sim \Gamma_2$ then $\phi_t(x_2) = \phi_{t+t_0}(x_1)$ which is equivalent to $\phi_{t-t_0}(x_1) = \phi_t(x_1)$, i.e., $\Gamma_2 \sim \Gamma_1$. Transitive: If $\Gamma_1 \sim \Gamma_2$ and $\Gamma_2 \sim \Gamma_3$ then $\phi_t(x_2) = \phi_{t+t_0}(x_1)$ and $\phi_t(x_3) = \phi_{t+t_1}(x_2) = \phi_{t+t_0+t_1}(x_1)$, thus $\Gamma_1 \sim \Gamma_3$. This equivalence relation partitions the set of solution curves of (1) into equivalence classes called trajectories.

4. $\omega(\Gamma)$ cannot consist of one limit orbit and two equilibrium points; in case (d) there are two different topological types given by the top two figures in Figure 4 in Section 3.3. (Also, see p. 129 in the appendix).

6. (a) Replacing x by $-x$ and y by $-y$ does not change the system.

(b) For $x = y = 0$, $\dot{x} = 0$, and $\dot{y} = 0$, so the z-axis is invariant and consists of three trajectories: the origin together with the positive and negative z-axes.

(c) Substituting the coordinates for the equilibrium points into the right-hand side of the system gives zero; for $\sigma > 0$ and $\rho > 1$, the linear part at the origin has two negative eigenvalues and one positive eigenvalue.

(d) $\dot{V}(\mathbf{x}) = -2\sigma[(\rho x - y)^2 + \beta z^2] < 0$ except on the line $z = 0$, $y = \rho x$; thus, for $0 < \rho < 1$, **0** is globally asymptotically stable.

PROBLEM SET 3.3

1. (a) $\dot{r} = r(1 - r^2)\sin\left[1/\sqrt{|1-r^2|}\right]$, $\dot{\theta} = 1$ and $\dot{r} = 0$ if $r = \sqrt{1 \pm \left(1/n^2\pi^2\right)}$. This defines a sequence of limit cycles $\Gamma_n^{\pm}$ which approach the cycle Γ on $r = 1$; the limit cycles $\Gamma_n^{\pm}$ are stable for n odd and unstable for n even.

(b) Similarly, $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2)\sin[1/(1 - r^2)] = 0$ if $r = \sqrt{1-(1/n\pi)}$, n a nonzero integer; Γ_n is stable for n odd and positive or n even and negative and Γ_n is unstable for n even and positive or odd and negative.

2. $\dot{\theta} = 1$ and $\dot{r} = r(1 - r^2)^2 = 0$ if $r = 1$ and $\dot{r} > 0$ for $r \neq 0$ or 1.

3. From the example in Section 1.5 we have a one-parameter family of cycles lying on the ellipses $x(t) = \alpha\cos 2t$, $y(t) = (\alpha/2)\sin 2t$ with parameter $\alpha \in (0, \infty)$ and period $T_\alpha = \pi$.

(b) $\dot{r} \equiv 0$ and $\dot{\theta} = r > 0$ for $r > 0$; by substitution into the system of differential equations, $x(t) = \alpha\cos\alpha t$, $y(t) = \alpha\sin\alpha t$ is a periodic solution with period $T_\alpha = 2\pi/\alpha$ for $\alpha \in (0, \infty)$.

4–6. Use the result of Problem 1 in Section 2.14 to show that the system is Hamiltonian and then use Theorem 2 in Section 2.14 to determine which critical points are saddles and which are centers. (Also, see p. 130 in the appendix.)

7. (a) $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2)(4 - r^2)$ has two limit cycles $\Gamma_1 : \gamma_1(t) = (\cos t, \sin t)^T$ and $\Gamma_2 : \gamma_2(t) = (2\cos t, 2\sin t)^T$; Γ_1 is stable and Γ_2 is unstable; $\mathbf{0}$, Γ_1, Γ_2 are the only limit sets of this system.

(b) $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2 - z_0^2)(4 - r^2 - z_0^2)$ for $z = z_0$; the spheres $S_1 : r^2 + z^2 = 1$ and $S_2 : r^2 + z^2 = 4$ are invariant; there is no attracting set; cf. Example 2 in Section 3.2.

8. $\dot{\theta} = 1$, $\dot{r} = r(1 - r^2)(4 - r^2)$, $z(t) = z_0e^t$; Γ_1 and Γ_2 are unstable, $W^s(\Gamma_1) = \{x \in \mathbf{R}^3 \mid z = 0, 0 < r < 2\}$, $W^u(\Gamma_1) = \{x \in \mathbf{R}^3 \mid r = 1\}$, $W^s(\Gamma_2) = \Gamma_2$, $W^u(\Gamma_2) = \{x \in \mathbf{R}^3 \mid 1 < r < \infty\}$. The unit cylinder is the only attracting set for this system.

9. Γ_2 is a stable periodic orbit $W^s(\Gamma_2) = \{x \in \mathbf{R}^3 \mid 1 < r < \infty\}$; the origin, Γ_2, the z-axis and the cylinder $r = 2$ are attracting sets for this system; and the origin and Γ_2 are the only attractors for this system.

PROBLEM SET 3.4

1. Substitution into the system of differential equations shows that $\gamma(t)$ is a periodic solution. Since $\nabla \cdot \mathbf{f}(\gamma(t)) = -2$ (since $1 - x^2/4 - y^2 = 0$ on $\gamma(t)$), it follows from the corollary to Theorem 2 that Γ is a stable limit cycle.

2. Substitution shows that $\gamma(t)$ is a periodic solution. In cylindrical coordinates $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$ and $\dot{z} = z$, which has the solution $\phi_t(r_0, \theta_0, z_0) = ([1 + (1/r_0^2 - 1)e^{-2t}]^{-1/2}, t + \theta_0, z_0e^t)^T$; thus $P(r_0, z_0) = ([1 + (1/r_0^2 - 1)e^{-4\pi}]^{-1/2}, z_0e^{2\pi})^T$, $DP(r_0, z_0) = \text{diag}\big[e^{-4\pi}r_0^{-3} \cdot [1 + (1/r_0^2 - 1)e^{-4\pi}]^{-3/2}, e^{2\pi}\big]$ and $DP(1, 0) = \text{diag}\,[e^{-4\pi}, e^{2\pi}] = e^{2\pi B}$ where $B = \text{diag}\,[-2, 1]$.

3. (a) For $\mathbf{x}_0 = (x_0, 0)$, $\phi_t(\mathbf{x}_0) = e^{at} R_{bt}\,\mathbf{x}_0 = e^{at}(x_0 \cos bt, x_0 \sin bt)^T$; at $t = 2\pi/|b|$, we get $P(x_0) = x_0 \exp[2\pi a/|b|]$; for $d(x) = P(x) - x = x \exp[2\pi a/|b|] - x$, $d'(0) = d'(x) = \exp[2\pi a/|b|] - 1$ and clearly $d(-x) = -d(x)$.

   (b) $P(s) = [1 + (1/s^2 - 1)e^{-4\pi}]^{-1/2}$ for $s \neq 0$ and $P(0) = 0$; and this is equivalent to $P(s) = s[s^2 + (1 - s^2)e^{-4\pi}]^{-1/2}$ which is (real) analytic for all $s \in \mathbf{R}$ since $s^2 + (1 - s^2)e^{-4\pi} = e^{-4\pi} + (1 - e^{-4\pi})s^2 > 0$ for all $s \in \mathbf{R}$; since $P'(s) = e^{-4\pi}[s^2(1 - e^{-4\pi}) + e^{-4\pi}]^{-3/2}$ for all $s \in \mathbf{R}$, $P'(0) = e^{2\pi}$ and $d'(0) = e^{2\pi} - 1 > 0$; thus, the origin is a simple focus which is unstable.

4. $\dot\theta = 1$, $\dot r = r(1 - r^2)^2$ and $\gamma(t) = (\cos t, \sin t)^T$ is a semi-stable limit cycle of this system; since $\nabla\cdot\mathbf{f}(\gamma(t)) \equiv 0$, it follows from Theorem 2 that $d(0) = d'(0) = 0$ and hence $k \geq 2$ in Definition 2, i.e., Γ is a multiple limit cycle.

5. If $a = 0$, $b \neq 0$, $a_{20} + a_{02} = b_{20} + b_{02} = 0$, then according to equation (3), $\sigma = d'''(0) = 0$ and therefore the first non-vanishing derivative $d^{(k)}(0) \neq 0$ has $k = 2m + 1 \geq 5$, i.e., the origin is either a center or a focus of multiplicity $m \geq 2$.

PROBLEM SET 3.5

1. Direct substitution shows that $\gamma(t)$ is a periodic orbit of the system. The linearization about $\gamma(t)$ has $A = D\mathbf{f}(\gamma(t)) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\Phi(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$ as its fundamental matrix satisfying $\Phi(0) = I$. It follows that $\Phi(t) = Q(t)e^{Bt}$ with $Q(t)$ given in Example 1 and $B = \mathrm{diag}\,[0, 0, -1]$; therefore, the characteristic exponents of $\gamma(t)$ are $\lambda_1 = 0$ and $\lambda_2 = -1$ and the characteristic multipliers are 1 and $e^{-2\pi}$; dim $S(\Gamma) = 2$, dim $C(\Gamma) = 2$ and dim $U(\Gamma) = 1$.

2. Direct substitution shows that $\gamma(t)$ is a periodic solution of period π; the linearization about $\gamma(t)$ has $A(t) = Df(\gamma(t)) = \begin{bmatrix} -2\cos^2 2t & -4-2\sin 4t & 0 \\ 1-(\sin 4t)/2 & -2\sin^2 2t & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and direct substitution into $\dot{\mathbf{x}} = A(t)\mathbf{x}$ shows that the given $\Phi(t)$ is a fundamental matrix for this non-autonomous linear system; $\Phi(t) = Q(t)e^{Bt}$ with $B = \text{diag}[-2, 0, 1]$ and $Q(t) =$

$\begin{bmatrix} \cos 2t & -2\sin 2t & 0 \\ \frac{1}{2}\sin 2t & \cos 2t & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The characteristic exponents of $\gamma(t)$ are $\lambda_1 = -2$ and $\lambda_2 = 1$ and the characteristic multipliers are $\exp(-2\pi)$ and $\exp(\pi)$; dim $S(\Gamma)$ = dim $U(\Gamma) = 2$ and dim $C(\Gamma) = 1$. The periodic orbit $\gamma(t)$ is an ellipse in the x,y plane; $W^u(\Gamma)$ is a vertical, elliptical cylinder through Γ, $W^s(\Gamma)$ is the x,y plane without the origin and $W^c(\Gamma) = \Gamma$.

4. (a,b) $\phi_t(\mathbf{x}_0) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 1-\cos t & \sin t & 1 \end{bmatrix} \mathbf{x}_0 \equiv \Phi(t)\mathbf{x}_0$ where for $\mathbf{u}(t, \mathbf{x}_0) = \phi_t(\mathbf{x}_0)$, $\Phi(t) = D\phi_t(\mathbf{x}_0) = \partial\mathbf{u}(t, \mathbf{x}_0)/\partial\mathbf{x}_0$.

(c) $\gamma(t)$ is the periodic solution through the point $(1, 0, 0)^T$ at $t = 0$, $A = Df(\gamma(t)) =$ $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and direct substitution shows that $\Phi(t)$ satisfies the linear differential equation (2) and $\Phi(0) = I$.

7. (a) $\frac{1}{3}\begin{bmatrix} 4e^t - e^{-2t} & 2(e^{-2t} - e^t) \\ 2(e^t - e^{-2t}) & 4e^{-2t} - e^t \end{bmatrix} = P\begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}P^{-1} = e^{B_1 t}$ according to Proposition 1 in Section 1.3 where $P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}$. Thus, $\Phi(t) = Q(t)e^{Bt}$ with $Q(t) =$ diag $[R_t, I_3]$ and $B =$ diag $[0, 0, B_1, 0]$ where I_3 is the 3 x 3 identity matrix; it follows that the characteristic exponents of Γ are 0, 0, 1, –2 and that dim $W^s(\Gamma) = 2$, dim $W^u(\Gamma) = 2$, and dm $W^c(\Gamma) = 3$.

(b) Similarly, $\Phi(t) = Q(t)e^{Bt}$ with $Q(t) = \text{diag}[R_{2t}, A(t), 1]$ and $B = \text{diag}\,[-3, -3, 3, 3, 0]$ where R_{2t} is a 2 x 2 rotation matrix and $A(t) = [1, 0; t, 0]$; it follows that dim $W^s(\Gamma) =$ dim $W^u(\Gamma) = 3$ and dim $W^c(\Gamma) = 1$.

8. Direct substitution shows that $\gamma(t)$ is a periodic solution of the given system and that the given fundamental matrix $\Phi(t)$ is a solution of $\dot{\mathbf{x}} = A(t)\mathbf{x}$ with $A(t) = Df(\gamma(t)) =$

$$\begin{bmatrix} -8a\cos^2 4t & -2 - 4a\sin 4t\cos 4t & a\cos 4t \\ 8 - 16a\sin 4t\cos 4t & -8a\sin^2 4t & 2a\sin 4t \\ 0 & 0 & \cos 4t - a^2 \end{bmatrix}$$

provided $\alpha(t)$ and $\beta(t)$ satisfy the nonhomogeneous, periodic system given in this problem (it is not necesssary to solve the system for $\alpha(t)$ and $\beta(t)$ in order to finish the problem); $\Phi(t) = Q(t)e^{Bt}$ with $B =$ diag $[-8a, 0, -a^2]$ and

$$Q(t) = \begin{bmatrix} \cos 4t & -1/2\sin 4t & \alpha(t)e^{1/4\sin 4t} \\ 2\sin 4t & \cos 4t & \beta(t)e^{1/4\sin 4t} \\ 0 & 0 & e^{1/4\sin 4t} \end{bmatrix}.$$

The characteristic exponents are $\lambda_1 = -8a$ and $\lambda_2 = -a^2$; the characteristic multipliers are $e^{-4a\pi}$ and $\exp(-a^2\pi/2)$; for $a > 0$ dim $W^s(\Gamma) = 3$ and dim $W^u(\Gamma) = 1$ while for $a < 0$, dim $W^s(\Gamma) =$ dim $W^u(\Gamma) = 2$.

PROBLEM SET 3.6

1. dim $W^c(\Gamma) = 4$; solving the w, Z equation for w leads to $1 - w =$ $\left[1 \pm \sqrt{1-h^2-h^2Z^2}\right]/\left(1+Z^2\right)$ and substituting this into the first equation following Figure 5 and simplifying leads to the result for T_h^2.

2. (a) The Hamiltonian $H(x, y, z, w) = (\beta x^2 + \beta y^2 + z^2 + w^2)/2$; thus, for $H = 1/2$, trajectories lie on the ellipsoidal surface $\beta(x^2 + y^2) + z^2 + w^2 = 1$ and for $h \in (0, 1)$ if $x^2 + y^2 = h^2/\beta$, then $z^2 + w^2 = 1 - h^2$, i.e., trajectories lie on the tori T_h^2.

   (b) As in Problem 1, we find $1 - w = \left[1 \pm \sqrt{1-h^2-h^2Z^2}\right]/\left(1+Z^2\right)$ and then $Z^2 + \beta(X - 1/h\sqrt{\beta})^2 = (1 - h^2)/h^2$.

   (c) According to Problem 2 in 3.2, the flow is dense in each of the tori T_h^2 if β is irrational; and it consists of a one-parameter family of periodic orbits if β is rational.

4. Under the projective transformation in Problem 3, Γ_0 gets mapped onto $Z = W = 0$, i.e., the Y-axis; Γ_1 gets mapped onto $Y = y/(k - x)$, $Z = 1/(k - x)$, $W = 0$ and then substitution into $H(x, y, z, w) = k^2/2$ leads to $(Z - 3k)^2 + 3Y^2 = 3(3k^2 - 1)$, the equation of an ellipse; the linearization about Γ_1 shows that Γ_1 has characteristic exponents $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$ and $\lambda_4 = -1$ and dim $W^s(\Gamma_1) =$ dim $W^u(\Gamma) = 2$ as in Section 3.5.

PROBLEM SET 3.7

1. (a) $\dot{r} = r^5 - 3r^3 + r$ and $\dot{\theta} = 1$ so $\dot{r} < 0$ on $r = 1$ and $\dot{r} > 0$ on $r = 2$; thus, the α-limit set of any trajectory that starts in A_1 is in A_1 and by the Poincaré-Bendixson theorem $\alpha(\Gamma)$ is a periodic orbit (since A_1 contains no critical points).

   (b) Since the eigenvalues of the linear part, $Df(\mathbf{0})$, are $\pm 2i$ and since $\dot{r} > 0$ for sufficiently small $r > 0$, the origin is an unstable focus; thus, the ω-limit set of any trajectory that starts in A_2 is in A_2, etc.

   (c) $r = \sqrt{3 - \sqrt{5}} / \sqrt{2}$ is a stable limit cycle and $r = \sqrt{3 + \sqrt{5}} / \sqrt{2}$ is an unstable limit cycle.

2. (a) $\dot{r} = r - r^3 (\cos^4\theta + \sin^4\theta) = r - r^3 + r^3 \sin^2 2\theta/2$; thus, $\dot{r} \leq r(1 - r^2/2) = (\sqrt{2} + \varepsilon)(-\sqrt{2}\,\varepsilon - \varepsilon^2/2) < 0$ for $r = \sqrt{2} + \varepsilon$ and $\varepsilon > 0$; $\dot{r} \geq r(1 - r^2) = (1 - \varepsilon)(2 - \varepsilon)\varepsilon > 0$ for $r = 1 - \varepsilon$ and $0 < \varepsilon < 1$. Thus, for $0 < \varepsilon < 1$, any trajectory Γ entering the annular region $A_\varepsilon = \{1 - \varepsilon < r < \sqrt{2} + \varepsilon\}$ at $t = t_0$ remains in A_ε for all $t > t_0$. Since there are no critical points in A_ε, it follows from the Poincaré-Bendixson theorem that there is a periodic orbit $\Gamma_0 = \omega(\Gamma) \subset A_\varepsilon$ and since this is true for all $\varepsilon \in (0, 1)$, it follows that $\Gamma_0 \subset \bar{A}$. Also, the only points on the circle $r = 1$ where a limit cycle could intersect $r = 1$ are the points on $r = 1$ where $\theta = 0, \pi/2, \pi, 3\pi/2$ (where $\dot{r} = 0$). But at all other points on $r = 1$, $\dot{r} > 0$; so it is impossible for any limit cycle to intersect $r = 1$. A similar argument shows that no limit cycle can intersect $r = \sqrt{2}$; thus, there is at least one limit cycle in the annular region A.

   (b) Using Dulac's Theorem, this analytic system has a finite number of limit cycles and since the boundary of A_ε is "incoming" for all $\varepsilon \in (0, 1)$, at least one of these limit cycles must be stable.

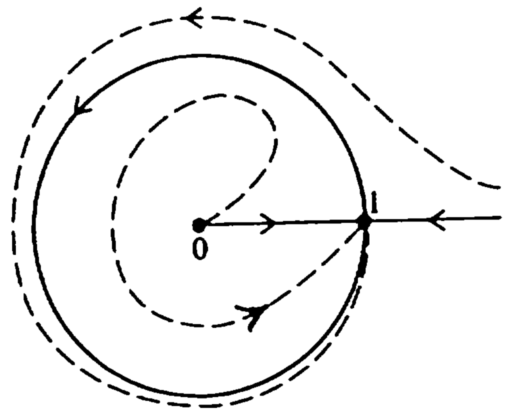
4. (a) The only two phase portraits that are topologically distinct in the annular region A are the two for which the cycles $r = 1$ and $r = 2$ have either the same or the opposite orientations; all other possible phase portraits are equivalent to one of these two under the homeomorphism of A given either by $H(x, y) = (x, -y)$ or $H(x, y) = 2/\sqrt{x^2 + y^2}$.

(b) As in the proof of Poincaré's theorem: On any ray $\theta = \theta_1$, we have $\dot{\theta} > 0$ on $r = 1$ and $\dot{\theta} < 0$ on $r = 2$ or vice versa; thus by the intermediate value theorem $\dot{\theta} = 0$ at some point $r \in (1, 2)$; then, by continuity, there is a closed curve Γ_0 of points on which $\dot{\theta} = 0$; i.e., on which the motion is radial; let Γ_1 be the curve $\phi_{t_1}(\Gamma_0)$ with $t_1 > 0$; then since the flow ϕ_t is area preserving, it follows that Γ_0 and Γ_1 must enclose the same area and therefore they must cross at least twice; if they cross exactly twice, at $\mathbf{x}_1$ and $\mathbf{x}_2$, then $\dot{r}$ must have the opposite sign on the arcs $\mathbf{x}_1\mathbf{x}_2$ and $\mathbf{x}_2\mathbf{x}_1$ of Γ_0 and therefore, by continuity, $\dot{r} = 0$ at $\mathbf{x}_1$ and $\mathbf{x}_2$ which are thus critical points of (1) since $\dot{r} = \dot{\theta} = 0$ there; if there are more points of intersection, it can still be shown that $\dot{r}$ changes sign at least twice.

6. (a) Even though the points x_n may not lie on a straight line, by Lemma 1 there is a transversal ℓ through the point $x_0 \in \Gamma_0$ and since $x_n \to x_0$, it follows from Lemmas 1 and 2 that Γ_n crosses ℓ exactly once at a point $\tilde{x}_n$ (for all sufficiently large n); thus, by Theorem 1 in Section 3.4, $\phi_{\tau(\tilde{x}_n)}(\tilde{x}_n) = \tilde{x}_n$, i.e., $\tau(\tilde{x}_n) = T_n$ and $\tau(x_0) = T_0$; therefore, by the continuity of τ, $T_n \to T_0$ as $n \to \infty$ since $\tilde{x}_n \to x_0$ as $n \to \infty$.

(b) See Figure 6 in Section 4.5

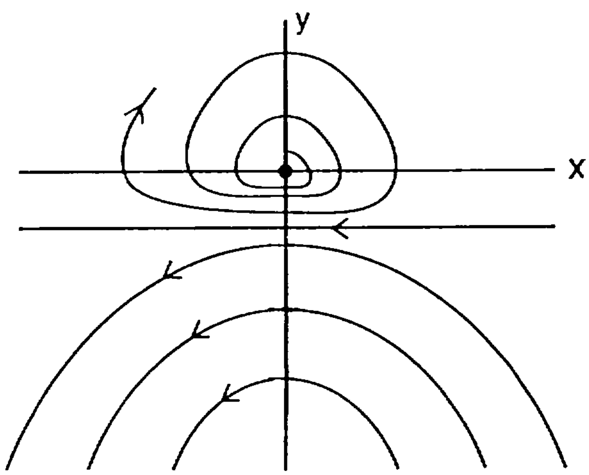
7. The phase portrait is given by the separatrix configuration shown here; and even though the critical point (1, 0) is the ω-limit set of every trajectory, except for the critical point at **0**, (1, 0) is not a stable critical point.

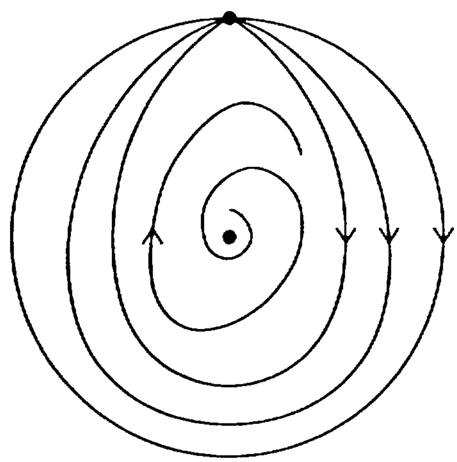


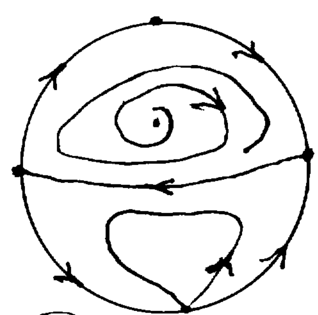
8. For $y = -1$, we have $\dot{y} = 0$ and $\dot{x} = -1$; i.e., the line $y = -1$ is a trajectory. Also, the origin is the only critical point of this system and since $Df(0, 0) = [0, 1; -1, 1]$, the origin is an unstable focus. Since $r\dot{r} = y^2(1 + y) / (1 + y^2) > 0$ for $y > -1$, there are no limit cycles around the origin in the half plane $y > -1$. Thus, according to the Poincaré–Bendixson Theorem for analytic systems, any trajectory starting on the positive y-axis has the separatrix cycle consisting of the trajectory $y = -1$ and the point at infinity on the Bendixson sphere as its ω-limit set; i.e., this system has an unbounded oscillation. Since

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{1+y}{1+y^2} - x\left(1 + \frac{1}{y}\right) \to -x$$

as $y \to -\infty$, it follows that any trajectory in the half plane $y < -1$ (which can be represented as a function $y(x)$ since $\dot{x} = y < -1$), satisfies $y(x) = y_0 - x^2/2 + 0(1/x)$ as $|x| \to \infty$, i.e., as $t \to \pm\infty$. The phase portraits on $\mathbf{R}^2$ and on the Bendixson sphere are given by the following figures:







Poincaré Sphere

PROBLEM SET 3.8

2. f has exactly two positive zeros at $a_{1,2} = \sqrt{5 \pm \sqrt{17}} / 2$, $f'(x) > 0$ for $x > \sqrt{5} / 2$ and hence for $x > a_2$, and the function $F(x) = .32x^5 - 4x^3/3 + .8x$ satisfies $F(a_1) > 0$ and $F(a_2) < 0$.

3. $a_1 = -72$, $a_3 = 392/3$, $a_5 = -224/5$ and $a_7 = 128/35$.

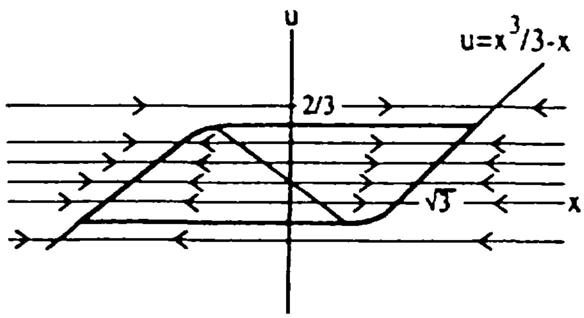
4. The system in Theorem 6 with $a_1 = 1152$, $a_3 = -6560/3$, $a_5 = 4368/5$, $a_7 = -768/7$, and $a_9 = 256/63$ (and all other a's = 0) has exactly four limit cycles asymptotic to $r = 1, 2, 3, 4$ as $\varepsilon \to 0$.

5. By the symmetry with respect to the y-axis, the critical point $(0, F(0))$ is a center. (Also, see p. 134 in the appendix.)

6. By Lienard's theorem (or its corollary in this section), van der Pol's system has a unique limit cycle and it is stable.

(a) By Theorem 6, the limit cycle is asymptotic to the circle $r = 2$ as $\mu \to 0$.

(b) Under the given transformation we get $x'(\tau) = \mu^2(u + x - x^3/3)$ and $u'(\tau) = -x$ and then with $t = \mu^2\tau$ and $\varepsilon = 1/\mu^2$, we get $\dot{x} = u + x - x^3/3$ and $\dot{u} = -\varepsilon x$. For $0 < \varepsilon << 1$ we have $0 < |\dot{u}| << 1$, $\dot{u} > 0$ for $x < 0$ and $\dot{u} < 0$ for $x > 0$; and since $\dot{x}$ changes sign on the curve $u = x^3/3 - x$, it follows that the limit cycle is approximated by the darkened curve shown in Figure 8 as $\varepsilon \to 0$, i.e., as $\mu \to \infty$. Cf. F. Dumortier's analysis of the "canard" phenomenon in Section 5 of his article in NATO Adv. Studies, C408 (1993) 20–73.



PROBLEM SET 3.9

1. If (1) has a separatrix cycle $S = \bigcup_{j=1}^{m} \Gamma_j \subset E$, then by Green's theorem with $R = \text{int}(S)$, $I = \iint_R \nabla \cdot \mathbf{f}\,dxdy = \int_S (Pdy - Qdx) = \sum_{j=1}^{m} \int_{-\infty}^{\infty} (P\dot{y} - Q\dot{x})dt = 0$ since $\dot{x} = P$ and $\dot{y} = Q$ on Γ_j; but if $\nabla \cdot \mathbf{f}$ is not identically zero and does not change sign in E, then $|I| > 0$, a contradiction; thus, there is no separatrix cycle S of (1) lying entirely in E.

2. Suppose there is a periodic orbit Γ lying in E and let $R = \text{int}(\Gamma)$, a simply connected region; then by Green's theorem $I = \iint_R \nabla \cdot (B\mathbf{f})dxdy = \int_\Gamma B(Pdy - Qdx) = \int_{-\infty}^{\infty} B(P\dot{y} - Q\dot{x})dt = 0$ and this leads to a contradiction as in Problem 1. Next, suppose there are two periodic orbits Γ_1 and Γ_2 in the annular region A and let Γ_0 be an arc from a point on Γ_1 to a point on Γ_2 which lies in the region between Γ_1 and Γ_2; then for $\Gamma = \Gamma_1 + \Gamma_0 - \Gamma_2 - \Gamma_0$, the simply connected region R = $\text{int}(\Gamma) \subset A$ and by Green's theorem $I = \iint_R \nabla \cdot (B\mathbf{f})dxdy = \int_{\Gamma_1} + \int_{\Gamma_0} - \int_{\Gamma_2} - \int_{\Gamma_0} B(Pdy - Qdx) = \int_{\Gamma_1} B(Pdy - Qdx) - \int_{\Gamma_2} B(Pdy - Qdx) = 0$ and this leads to a contradiction as above.

3. (a) $\nabla \cdot \mathbf{f} = 1 - 3r^2 < 0$ for $r > 1/\sqrt{3}$; there is no contradiction to Bendixson's theorem because A is not a simply connected region.

   (b) From Problem 2 in Section 7, there is at least one limit cycle in the region A and then by Theorem 2 (with B = 1) there is at most one limit cycle in A.

4. (a) Since $\nabla \cdot \mathbf{f} = 1 - x^2 > 0$ for $|x| < 1$, it follows from Bendixson's criteria that any limit cycle of van der Pol's equation must cross both of the lines $x = \pm 1$. (Note that van der Pol's system is invariant under $x \to -x$ and $y \to -y$ and therefore any limit cycle of this system is symmetric with respect to the origin.)

(b) First of all, if f has no positive zeros then $F'(x) = f(x)$ is either increasing or decreasing and therefore by the corollary to Theorem 3, there is no limit cycle of (1); if x_1 is the smallest zero of f then since $\nabla\cdot\mathbf{f} = -F'(x) = -f(x) \neq 0$ for $|x| < x_1$, it follows as in part (a) that any limit cycle of (1) must cross $x = x_1$ and $x = -x_1$.

5. (b) Change the time scale to obtain $\dot{x} = y$, $\dot{y} = -x + y(1 + x^2 + x^4)$ and then compute $\nabla\cdot\mathbf{f}$ (or $\dot{r}$) for this system to show that it has no limit cycle in $\mathbf{R}^2$.

PROBLEM SET 3.10

1. (a) The stable manifold is tangent to the stable subspace $E^s = \{(y, z) \in \mathbf{R}^2 \mid y = 0\}$ at $\mathbf{0}$; and since for $y = 0$, $\dot{y} = -5z^2 < 0$, the stable manifold is as shown in Figure 6.

(b) Since $\dot{y} < 0$ for $y = 0$, the separatrix Γ having $(1, 0, 0)$ as its ω-limit point approaches $(1, 0, 0)$ through points where $y > 0$; since there are no critical points in the finite plane, there are no cycles there and hence, by the generalized Poincaré-Bendixson theorem, $\alpha(\Gamma)$ is a critical point on the equator of the Poincaré sphere with $y > 0$; it cannot be the stable node at $(1, 2, 0)/\sqrt{5}$; so it must be the unstable node at $(-1, 2, 0)/\sqrt{5}$.

2. By the theorem in Section 1.5, $\mathbf{0}$ is an unstable focus for the linear part of this system at $\mathbf{0}$; by Theorem 4 in Section 2.10 (or by the Hartman Grobman theorem and Theorem 2 in 2.10), $\mathbf{0}$ is an unstable focus for the nonlinear system as well; and since $\dot{x} = -y$ for $x = 0$, $\dot{x} < 0$ if $y > 0$ and $\dot{x} > 0$ if $y < 0$; therefore, the flow swirls counterclockwise around the focus at $\mathbf{0}$. Since $\mathbf{0}$ is unstable and there are no other critical points in $\mathbf{R}^2$, it follows from the generalized Poincaré-Bendixson theorem that the (unique) stable limit cycle established in Section 3.8 is the ω-limit set of every trajectory, except the equilibrim point $\mathbf{0}$.

3. (a) We have an unstable node at the origin; stable nodes at $(\pm 1, 0, 0)$; and saddles at $(0, \pm 1, 0)$.

(c) We have an unstable focus at the origin, with a counterclockwise flow; there is a cycle on the equator of the Poincaré sphere and no limit cycles in $\mathbf{R}^2$ for this linear system.

4. You should determine that (4, 2) is an unstable node and (−2, −1) is a stable focus for this nonlinear system. (Cf. Sections 1.5 and 2.10.) Also, determine that there is a stable node at (0, 1, 0), an unstable node at (0, −1, 0), and saddles at $\pm(\sqrt{2}, 1, 0)/\sqrt{3}$ and at $\pm(\sqrt{2}, -1, 0)/\sqrt{3}$ on the equator of the Poincaré sphere. Note that $\dot{y} \leq 0$ on $y = 0$ and that $\nabla \cdot \mathbf{f} = 10y < 0$ for $y < 0$; so there are no limit cycles. The global phase portrait is determined by Figure 12(i).

5. You should determine that (0, 0) is an unstable node and that (0, −2) and $(\pm\sqrt{3}, 1)$ are saddles. (Cf. Sections 1.5 and 2.10.) Also, determine that there are stable nodes at (0, 1, 0) and $(\pm\sqrt{3}, -1, 0)/2$, and that there are unstable nodes at (0, −1, 0) and $(\pm\sqrt{3}, 1, 0)/2$ on the equator of the Poincaré sphere. Note that the y-axis consists of trajectories and that $\dot{y} \leq 0$ for $y = 0$. The global phase portrait is determined by Figure 12(vii).

6. You should determine that there is a center at (0, 1) (using the symmetry with respect to the y-axis) and that there is a saddle at (0, −1). (Cf. Sections 1.5 and 2.10.) Also, determine that there are nodes at the critical points $(\pm 1, 0, 0)$ on the equator of the Poincaré sphere. Use the symmetry with respect to the y-axis to aid in drawing the global phase portrait which is determined by Figure 12(v).

7. You should determine that there are centers at $(0, \pm 1)$ using the symmetry with respect to the y-axis (also, this system is Hamiltonian) and that there are saddles at $(\pm 1, 0)$. Also, you should determine that there are nodes at the critical points $(\pm 1, 0, 0)$ on the equator of the Poincaré sphere. Use the symmetries to aid you in drawing the global phase portrait which is given by Figure 12(vi).

8. You should determine that $(1, 0)$ is an unstable node and that $(-1, 0)$ is a saddle for the nonlinear system. (Cf. Sections 1.5 and 2.10.) Also, using Theorem 1 in Section 2.11, determine that there are saddle-nodes at the four critical points $(\pm 1, \pm 1, 0)/\sqrt{2}$ and also that there are nodes at the critical points $(\pm 1, 0, 0)$ on the equator of the Poincaré sphere. Determine that the x-axis consists of trajectories and that the global phase portrait is given by Figure 12(ii).

9. (a) The equation of the tangent plane to a level surface $F(\mathbf{x}) = 1$ is given by $\nabla F(\mathbf{x}_0)\cdot(\mathbf{x} - \mathbf{x}_0) = 0$ at a point $\mathbf{x}_0$ on the surface and this leads to $\mathbf{x}\cdot\mathbf{x}_0 = 1$.

(b) At each point $\mathbf{x} \in S^2$, $\mathbf{f}(\mathbf{x})$ must satisfy $\mathbf{f}(\mathbf{x})\cdot\mathbf{x} = 0$.

(c) Follow the hint to obtain $\mathbf{f}$ and then show that $\mathbf{f}(\mathbf{x})\cdot\mathbf{x} = 0$.

10. $a \leftrightarrow i$, $b \leftrightarrow vii$, $c \leftrightarrow v$, $d \leftrightarrow vi$, $e \leftrightarrow ii$ as determined in Problems 4–8.

11. $i \leftrightarrow a$, $ii \leftrightarrow c$, $iii \leftrightarrow d$, $iv \leftrightarrow g$, $v \leftrightarrow e$, $vi \leftrightarrow f$, $vii \leftrightarrow b$.

12. The homoclinic loop at the saddle point $(0, -1)$ is given by $(x^2 + y^2 - 2y + 1)e^y = 4/e$.

13. (a) This follows directly by converting to polar coordinates since $\dot{r} = (x\dot{x} + y\dot{y})/r = r(x^2 - y^2)$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2 = 2xy$.

(b) Substituting $x = \xi/\rho^2$ and $y = \eta/\rho^2$ into the equation obtained in part (a) yields $d\rho/d\theta = -\rho(\xi^2 - \eta^2)/2\xi\eta$ which can be written as $\dot{\rho} = (\xi^2 - \eta^2)/\rho$, $\dot{\theta} = -2\xi\eta/\rho^2$ and therefore $\dot{\xi} = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta = \xi$ and $\dot{\eta} = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta = -\eta$.

(c) The flow on S^2 in Figure 4 follows exactly as in Example 1 and then projecting from the north pole of S^2 onto the ξ, η plane at the south pole of S^2 yields the flow in Figure 14(a); the "blow-up," shown in 14 (a), reduces the complicated critical point at the origin to four hyperbolic critical points (nodes) on the unit circle; shrinking this circle to the point at the origin yields Figure 14(b).

PROBLEM SET 3.11

1. In Figure 5 there are four (parabolic) strip regions; in Figure 7 there are three (parabolic) strip regions; and in Figure 9 there is one (hyperbolic) strip region and one spiral region.

2. You should determine that (0, 2) is a stable node and that (1, 0) is a saddle for the nonlinear system. (Cf. Sections 1.5 and 2.10.) You should also determine that according to Theorem 1 in Section 2.11, there are saddle-nodes at (±1, 0, 0) and according to Theorem 2 in Section 2.11, there are critical points with an elliptic domain at (0, ±1, 0). Finally, you should determine that the phase portrait is given by Figure 12(iv) in Section 3.10.

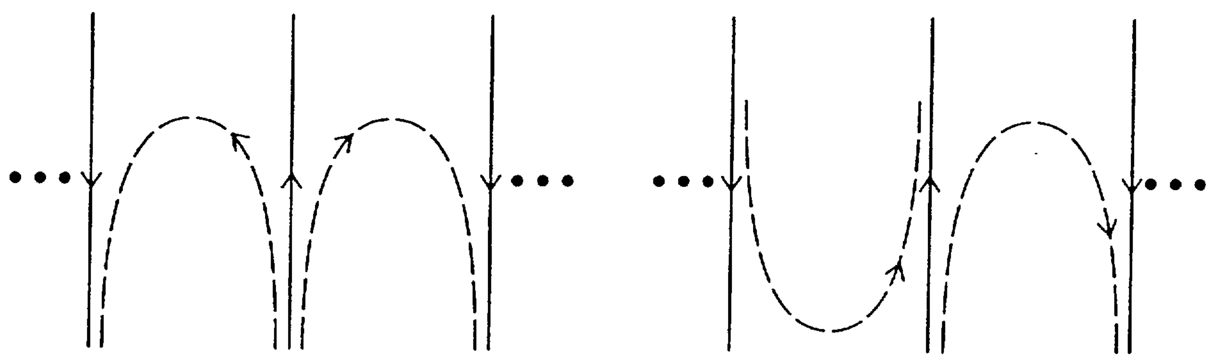
3. You should determine that (2, 2) is a stable node, that (–1, –1) is an unstable focus, and that (0, –2) is a saddle for the nonlinear system. (Cf. Sections 1.5 and 2.10.) You should also determine that according to Theorem 2 in Section 2.11 there are saddles at $\pm(1, 2, 0)/\sqrt{5}$ and a critical point with an elliptic sector at (0, 1, 0) and (0, –1, 0) on the equator of the Poincaré sphere. Finally, you should determine that the phase portrait is given by Figure 12(iii) in Section 3.10.

4. For $\alpha > 0$ you should determine that (0, 0) is an unstable node and that $(-\alpha, 0)$ is a saddle for the nonlinear system. Then determine that there are stable nodes at (0, ±1, 0) and using Theorem 1 in Section 2.11 determine that there are saddle-nodes at (±1, 0, 0). Draw the global phase portrait using the fact that the x and y axes and the straight line $x = -\alpha$ consist of trajectories. For the bifurcation value $\alpha = 0$, determine that there is a saddle-node at (0, 0) and that for $\alpha < 0$, the saddle and the node have interchanged their relative positions from those with $\alpha > 0$.

6. You should determine that there is an unstable node at the origin and stable nodes at (0, a/b) and (a/b, 0) as well as a saddle at $(r - b, s - b)a/(rs - b^2)$ for the nonlinear system. (Cf. Sections 1.5 and 2.10.) Also, determine that there are saddles at (±1, 0, 0) and (0, ±1, 0) and nodes at $\pm(r - b, s - b, 0)/\sqrt{(r-b)^2 + (s-b)^2}$ on the equator of the Poincaré sphere.

Use the fact that the x and y axes and the straight line $y(r - b) = x(s - b)$ consist of trajectories to aid in drawing the global phase portrait. If the initial numbers of the two competing species x_0 and y_0 (are positive and) lie on the above-mentioned straight line, then $x(t) \to a(r - b)/(rs - b^2)$ and $y(t) \to a(s - b)/(rs - b^2)$ as $t \to \infty$; but the probability of choosing a point in the first quadrant on that line is zero.

7. According to Definition 1′, the separatrices consist of the straight lines $x = n\pi$, $n = 0, \pm 1, \ldots$, since any other trajectory of the system can be embedded in a parallel region N with two other (curved) trajectories Γ_1 and Γ_2 satisfying the definition while the straight line trajectories $x = n\pi$ cannot. The flow is described by the following phase portraits:



PROBLEM SET 3.12

1. $$I_f(C) = (1/2\pi)\int_C (xdy - ydx)/(x^2 + y^2) = (1/2\pi)\int_0^{2\pi}(\cos^2\theta + \sin^2\theta)d\theta = 1 = I_g(C).$$

$$I_h(C) = (1/2\pi)\int_C (-ydx + xdy)/(x^2 + y^2) = (1/2\pi)\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)d\theta = 1.$$

$$I_k(C) = (1/2\pi)\int_C (-xdy + ydx)/(x^2 + y^2) = (1/2\pi)\int_0^{2\pi}(-\cos^2\theta - \sin^2\theta)d\theta = -1.$$

3. Let $\mathbf{v}_s = (1 - s)\mathbf{v} + s\mathbf{w}$ for $0 \le s \le 1$. Then $\mathbf{v}_0 = \mathbf{v} \ne \mathbf{0}$ and $\mathbf{v}_1 = \mathbf{w} \ne 0$. Suppose that for some $s \in (0, 1)$, $\mathbf{v}_s = 0$; this implies that $\mathbf{w} = [-(1 - s)/s]\mathbf{v}$; i.e., that $\mathbf{w}$ and $\mathbf{v}$ have the opposite directions; and that contradicts the hypotheses of Lemma 2. Next, for $s \in [0, 1]$,

$\mathbf{v}_s$ is continuous since the sum of the two continuous functions $(1 - s)\mathbf{v}$ and $s\mathbf{w}$ is continuous. Finally, it follows from the continuity of $\mathbf{v}_s$ with respect to s for $s \in [0, 1]$ and the fact that $I_{\mathbf{v}_s}(C)$ is an integer that $I_{\mathbf{v}_0}(C) = I_{\mathbf{v}_1}(C)$; otherwise there would exist a point $s^* \in [0, 1]$ at which the value of $I_{\mathbf{v}_s}(C)$ jumps by at least one unit; but this would contradict the continuity of $I_{\mathbf{v}_s}(C)$ with respect to s.

4. In Section 2.11, Figure 2 has $e = h = 1$ and therefore $I_f(\mathbf{0}) = 1$; Figure 3 has $h = 2$, $e = 0$ and $I_f(\mathbf{0}) = 0$; Figure 4 has $h = 2$, $e = 0$ and $I_f(\mathbf{0}) = 0$; and Figure 5 has $h = 0$, $e = 2$ and $I_f(\mathbf{0}) = 2$.

5. (a) From Figure 10, $T = 12$, $v = 6$, $\ell = 18$ and thus $\chi(T^2) = T + v - \ell = 0$.

   (b) Since $p = 1$ for the Klein bottle K, $\chi(K) = 2(1 - p) = 0$ and since the uniform parallel flow $\mathbf{f}$ shown in Figure 11 has no critical points, $I_{\mathbf{f}}(K) = 0 = \chi(K)$.

6. By Theorem 7 a saddle-node (sn) has index 0 and a critical point with an elliptic domain (ed) has index 1. Thus, in Section 3.10, Figures 4 and 7 have 2 saddles (s) and 4 nodes (n) on S^2 and thus $I(S^2) = 2$; Figure 9 has two foci (f) on S^2 and 2 sn on the equator of S^2 and thus $I(S^2) = 2$; Figure 12(i) has 4 n on S^2, 2 n and 4 s on the equator of S^2 and thus $I(S^2) = 2$; Figure 12(ii) has 2 s and 2 n on S^2, 2 n and 4 sn at infinity and thus $I(S^2) = 2$; Figure 12(iii) has 2 f, 2 s and 2 n on S^2 and 2 s and 2 ed at infinity and thus $I(S^2) = 2$; Figure 12(iv) has 2 s and 2 n on S^2 and 2 s and 2 ed at infinity and thus $I(S^2) = 2$; Figure 12(v) has 2 s and 2 centers (c) on S^2 and 2 n at infinity and thus $I(S^2) = 2$; Figure 12(vi) has 4 s and 4 c on S^2 and 2 n at infinity and thus $I(S^2) = 2$; Figure 12(vii) has 2 n and 6 s on S^2 and 6 n at infinity and thus $I(S^2) = 2$. Also, Figure 5 in Section 3.10 has 1 s in $\mathbf{R}^2$ and 2 n at infinity and thus $I(P) = 1$. In Section 3.10, Problem 3(a) has 1n in $\mathbf{R}^2$, 1 n and 1 s at infinity and thus $I(P) = 1$; Problem 3(b) has an infinite number of nonisolated critical points at infinity and we cannot compute $I(P)$; and Problem 3(c) has 1 f on $\mathbf{R}^2$ and no critical points at infinity and thus $I(P) = 1$.

7. The indices are 1, –2 and 2 respectively.

8. (a) There is a flow on S^2 with (i) one critical point which has two elliptic sectors and $I(S^2) = 2$; (ii) two nodes and $I(S^2) = 2$; and (iii) one critical point which has two elliptic sectors and two saddle-nodes in which case $I(S^2) = 2$.

(b) There is a flow on T^2 with (i) no critical points, i.e., a winding around T^2, and thus $I(T^2) = 0$; (ii) one critical point and a homoclinic loop at that critical point which then has two hyperbolic sectors and thus (by Bendixson's index theorem), $I(T^2) = 0$.

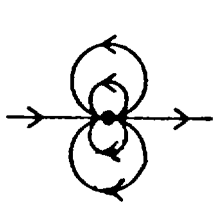
(c) There is a flow on the anchor ring with two saddles and $I = -2$.

(d) There is a flow on the double anchor ring (which has $p = 3$) with four saddles and $I = -4$.

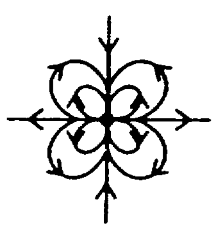
(e) A flow with a center at the origin and a cycle at infinity (such as $\dot{x} = y$, $\dot{y} = -x$) describes a flow on the projective plane, P, with one critical point; and a flow with a saddle node and a node in $\mathbf{R}^2$ and a cycle at infinity describes a flow on P with two critical points.

(f) Similar to the parallel flow on the rectangle, R, in Figure 11, you can describe a flow on R with a saddle-node which (as in Figure 11) would describe a flow on the Klein bottle with one critical point.

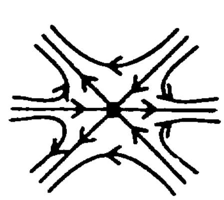
9. You should find $\dot{x} = r^k \cos k\theta$ and $\dot{y} = r^k \sin k\theta$ and therefore $\dot{r} = r^k \cos(k-1)\theta$ and $\dot{\theta} = r^{k-1}\sin(k-1)\theta$. For $k = 1$ there is a proper node at $\mathbf{0}$; for $k = 2$ there are two ray solutions at $\theta = 0$ and $\theta = \pi$ and two elliptic sectors, hence $I = 2$; for $k = 3$ there are four ray solutions, $\theta = 0, \pi/2, \pi, 3\pi/2$ and four elliptic sectors, hence $I = 3$; in general there are $2(k-1)$ ray solutions and elliptic sectors and hence $I = 1 + (k-1) = k$. The second equation yields $\dot{r} = r^k \cos(k+1)\theta$ and $\dot{\theta} = r^{k-1}\sin(k+1)\theta$; there are $2(k+1)$ ray solutions and hyperbolic sectors and hence $I = 1 - (k+1) = -k$. The phase portraits for $k = 2$ and 3 in these two equations are given by



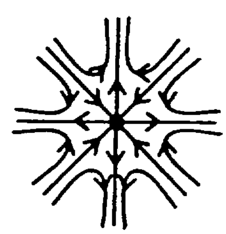
k = 2



k = 3



k = 2



k = 3

4. NONLINEAR SYSTEMS: BIFURCATION THEORY

PROBLEM SET 4.1

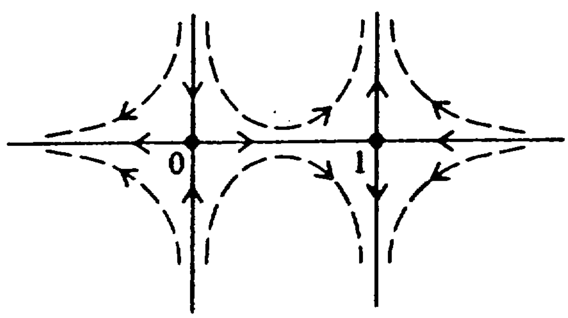
1. (a) Since $\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x}) = -\mu\mathbf{x}$, $D\mathbf{x} = I$ and $\|I\| = 1$, $\|\mathbf{f} - \mathbf{g}\|_1 = |\mu|(\max|\mathbf{x}| + 1)$.

   (b) The first system has a center at the origin and therefore $\lim_{t\to\infty} \phi_t(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$. For $\mu \neq 0$, for example if $\mu < 0$, then the second system has a sink at the origin and therefore $\lim_{t\to\infty} \psi_t(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}$. But if there were a homeomorphism H and a strictly increasing $\tau : \mathbf{R} \to \mathbf{R}$ such that $\phi_t = H^{-1}\psi_\tau H$, then it would follow that $\lim_{t\to\infty} \phi_t(1, 0) = H^{-1} \lim_{t\to\infty} \psi_t(H(1, 0)) = \mathbf{0}$, a contradiction; the case $\mu > 0$ is treated in the same way by considering $t \to -\infty$; thus, the two systems are not topologically equivalent for $\mu \neq 0$.

3. (a) $\det Df(\mathbf{0}) = -1$ and therefore the origin is a saddle for the nonlinear system; $\det Df(\pm 1, 0) = 2$, trace $Df(\pm 1, 0) = \mu$ and therefore $(\pm 1, 0)$ are stable (or unstable) foci for $\mu < 0$ (or $\mu > 0$); actually, they are foci for $|\mu| < \sqrt{8}$ and $\mu \neq 0$ while they are nodes for $|\mu| > \sqrt{8}$. (Cf. Sections 1.5 and 2.10.) Also, for $\mu \neq 0$, $\nabla\cdot\mathbf{f}(\mathbf{x}) = \mu$ does not change sign and Bendixson's criteria imply that there are no cycles (or separatrix cycles) in $\mathbf{R}^2$.

   (b) As in Problem 1, the fact that $\phi_t(\sqrt{2}, 0) \to \mathbf{0}$ and $\psi_t(\sqrt{2}, 0) \to (1, 0)$ as $t \to \infty$ (for $\mu < 0$) can be used to show that for arbitrarily small $|\mu| \neq 0$ the system in Example 3 is not topologically equivalent to the system with $\mu = 0$.

4. (a)



   (b) It can be shown that $\|\mathbf{f} - \mathbf{g}\|_1 = |\mu|\left(\max_{\mathbf{x}\in K}|\mathbf{x}| + 1\right)$. Thus, $\mathbf{f}$ and $\mathbf{g}$ are C^1-close on any compact K for sufficiently small $|\mu| \neq 0$. For the flow in (a), $|\phi_t(.5, 0)| \to 1$ as $t \to \infty$ and for the flow in (b), $|\psi_t(.5, 0)| \to \infty$ as $t \to \infty$ for $\mu \neq 0$ and this can be used to show that the two flows are not topologically equivalent as in Problem 1.

6(a)

5. According to the Corollary to Peixoto's theorem, only (vii) is structurally stable on S^2; (i, ii, iv, vii) are structurally stable on $\mathbf{R}^2$ (under strong C^1-perturbations) according to Theorem 4; (iii) is not structurally stable on $\mathbf{R}^2$ (under strong C^1-perturbations) according to Theorem 4 since it has a saddle connection between a saddle in $\mathbf{R}^2$ and a SAI, however it is structurally stable on any bounded region of $\mathbf{R}^2$; and (v, vi) are not structurally stable on bounded regions in $\mathbf{R}^2$ since they have saddle connections and nonhyperbolic critical points in $\mathbf{R}^2$.

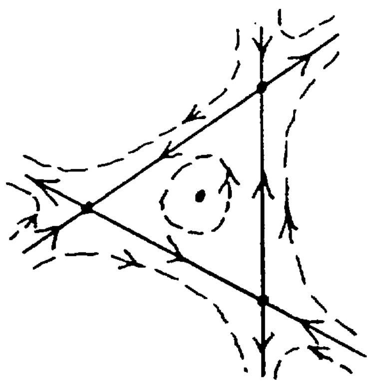
6. (a) There is only one critical point (0,0), a stable node, which is hyperbolic and there is no SAI; therefore, the system is structurally stable on R^2 (with respect to the C^1 strong topology) by Theorem 4. (It is not s.s. on S^2 by Cor. 1, but it is s.s. on any bounded region of R^2 which is what G/H ask for on p. 42).

(b) There is a nonhyperbolic critical point at (0,0) and the system is therefore not Structurally stable by Theorem 4.

(c) There is a nonhyperbolic critical point at (0, 0) and also saddle connections and therefore the system is not structurally stable by Theorem 4.

(d) There is a nonhyperbolic critical point (with $\lambda = \pm i$) at the origin and therefore the system is not structurally stable by Theorem 4. (Note that, by symmetry, the origin is a center for this system.)

7. This system is structurally unstable since there is a nonhyperbolic critical point (with $\lambda = \pm i$) at the origin; and also since there are saddle connections.

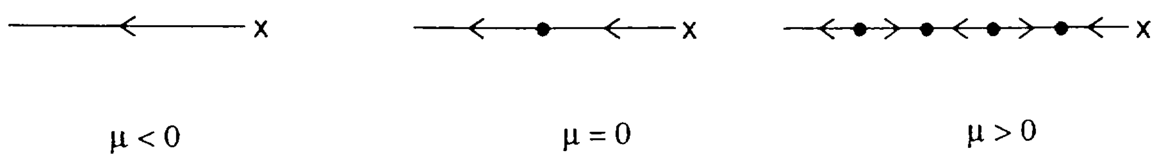


8. For Example 2 with $\mu < 0$, $\Omega = \{\mathbf{0}\}$; with $\mu = 0$, $\Omega = \{\mathbf{0}, \Gamma\}$ where Γ is the semistable limit cycle; and with $\mu > 0$, $\Omega = \{\mathbf{0}, \Gamma_1, \Gamma_2\}$ where Γ_1 and Γ_2 are the limit cycles shown in Figure 2. For Example 3 with $\mu < 0$, $\Omega = \{(0, 0), (\pm 1, 0)\}$ with $\mu = 0$, $\Omega = \mathbf{R}^2$; and with $\mu > 0$, $\Omega = \{(0, 0), (\pm 1, 0)\}$.

10. Ω consists of the set of critical points, limit cycles and graphics in each of the phase portraits in Figure 5.

PROBLEM SET 4.2

1. The critical points are at $x = \pm 2\sqrt{\mu}$ and $x = \pm\sqrt{\mu}$ for $\mu \geq 0$. The bifurcation value is $\mu = 0$. And the phase portraits are given by

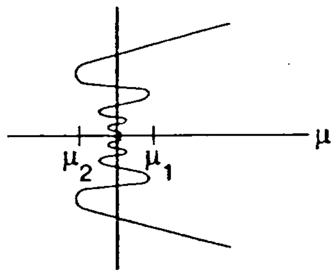


2. The critical points are at $x = 0$ and $x = \mu^2$; and the bifurcation value is $\mu = 0$.

3. (a) $f'(x) = 3x^2 \sin(1/x) - x\cos(1/x)$ for $x \neq 0$ and $f'(0) = \lim_{x \to 0} x^2 \sin(1/x) = 0$; since $\lim_{x\to 0} f'(x) = 0 = f'(0)$, f' is continuous at $x = 0$ and also for all $x \neq 0$. (a) For $\mu = 0$ there are critical points at $x = 1/n\pi$, $n = \pm 1, \pm 2, \cdots$ and at $x = 0$; since $f'(1/n\pi) = -(-1)^n/n\pi \neq 0$, the nonzero critical points are hyperbolic and alternate in stability; and since $f'(0) = 0$, the origin is a nonhyperbolic critical point.

(b) $\mu = 0$ is a bifurcation value since there are an infinite number of critical points in that case.

(c) From the bifurcation diagram, it can be seen that there are an infinite number of saddle-node bifurcations (at the points μ_n with vertical tangent lines) and that they accumulate at $\mu = 0$.

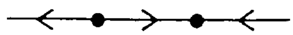


4. $\mathbf{x}_0 = (0, 0)$, $\mu_0 = 0$, $A = A^T = [0, 0; 0, -1]$ and the corresponding eigenvectors $\mathbf{v} = \mathbf{w} = (1, 0)^T$. Since $\mathbf{f}_\mu(\mathbf{x}, 0) = (x, 0)^T D\mathbf{f}_\mu(\mathbf{x}, 0) = [1, 0; 0, 0]$ and $D^2\mathbf{f}(\mathbf{x}, \mu) = [-2, 0, \cdots, 0]$, it follows that $\mathbf{w}^T\mathbf{f}_\mu(\mathbf{0}, 0) = 0$, $\mathbf{w}^T D\mathbf{f}_\mu(\mathbf{0}, 0)\mathbf{v} = 1$ and $\mathbf{w}^T D^2\mathbf{f}(\mathbf{0}, 0)(\mathbf{v}, \mathbf{v}) = -2$; thus the conditions (3) are satisfied. For $\mu = 0$, $\dim W^s(\mathbf{0}) = 1$, $\dim W^u(\mathbf{0}) = 0$ and $\dim W^c(\mathbf{0}) = 1$.

5. Similar to Problem 4 except that $D^2\mathbf{f}(\mathbf{x}, \mu) = [-6x, 0, \cdots, 0]$, $D^2\mathbf{f}(\mathbf{0}, 0) = 0$, $D^3\mathbf{f}(\mathbf{x}, \mu) = [-6, 0, \cdots, 0]$ and it follows that $\mathbf{w}^T\mathbf{f}_\mu(\mathbf{0}, 0) = 0$ and $\mathbf{w}^T D\mathbf{f}_\mu(\mathbf{0}, 0)\mathbf{v} = 1$ as above, but $D^2\mathbf{f}(\mathbf{0}, 0) = 0$ and $\mathbf{w}^T D^3\mathbf{f}(\mathbf{0}, 0)(\mathbf{v}, \mathbf{v}, \mathbf{v}) = -6$. For $\mu = 0$ $\dim W^s(\mathbf{0}) = \dim W^c(\mathbf{0}) = 1$ and $\dim W^u(\mathbf{0}) = 0$.

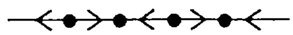
Problem Set 4.3

1. (a) Let $A \subset \mathbf{R}^3$ be the region above all the surfaces in Figure 3, $B \subset \mathbf{R}^3$ the region below all the surfaces in Figure 3, and $C \subset \mathbf{R}^3$ the shaded region in Figure 3. We then have

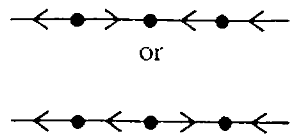


A

SN_2

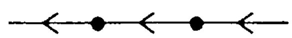


C

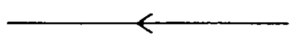


SN_1^+

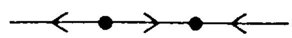
SN_1



$SN_1 \cap SN_1^+$



B

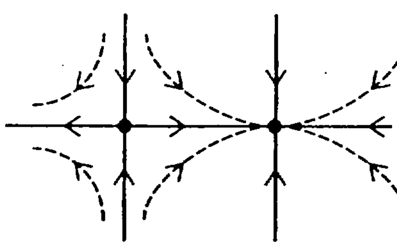


C_1 or C_2

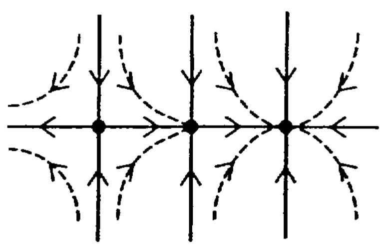
0

where SN_1^+ denotes the two pieces of the SN_1 surface that bound part of the shaded region in Figure 3.

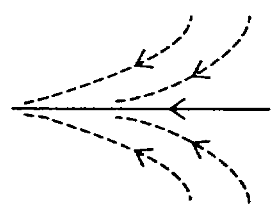
(b) The phase portraits for the given system in $\mathbf{R}^2$ follow from those in part (a) and $\dot{y} = -y$; for example, we have



A

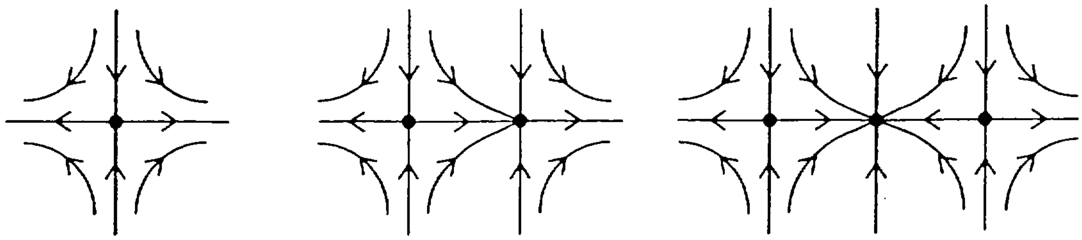


SN_2



B

2. As in Problem 3 in Section 2.12, the flow on the center manifold, $W^c(\mathbf{0}) : y = x^2 + \cdots$, is determined by $\dot{x} = x^3 + 0(x^4)$. This system therefore reduces to the normal form $\dot{x} = x^3$, $\dot{y} = -y$ and as in Example 1 of the cusp bifurcation in $\mathbf{R}^2$, we have the universal unfolding $\dot{x} = \mu_1 + \mu_2 x + x^3$, $\dot{y} = -y$ for this normal form or for the given system. For $0 < |\boldsymbol{\mu}| << 1$ and $\mu_1 < -\sqrt{4\mu_2^3/27}$, $\mu_1 = -\sqrt{4\mu_2^3/27}$ and $-\sqrt{4\mu_2^3/27} < \mu_1 < \sqrt{4\mu_2^3/27}$ respectively, we have:



3. As in Problem 4 in Section 2.12, the flow on the center manifold is determined by $\dot{x} = x^2 + 0(x^3)$. We therefore have the normal form $\dot{x} = x^2$, $\dot{y} = -y$ for this system. The universal unfolding is given as in equation (4), i.e., $\dot{x} = \mu + x^2$, $\dot{y} = -y$ and the various phase portraits are given in Figure 7 in Section 4.2 (with $x \to -x$).

4. The flow on $W^c(\mathbf{0})$ is determined by $\dot{x} = -x^4 + 0(x^5)$ and the universal unfolding as in equation (8) is given by $\dot{x} = \mu_1 + \mu_2 x + \mu_3 x^2 - x^4$, $\dot{y} = -y$. The various phase portraits can easily be determined from those in Problem 1.

6. (a) If $a \neq 0$ then the system (after normalizing the time) has the normal form $\dot{x} = x^2 + 0(x^3)$ on the center manifold; cf. Problem 6 in Section 2.12. And, as in Problem 3, the universal unfolding is given by $\dot{x} = \mu_1 + x^2$, $\dot{y} = -y$. It follows that for $a \neq 0$, this system has a codimension-one saddle-node bifurcation at $\mu_1 = 0$.

(b) If $a = 0$ and $bd \neq 0$ then the system (after rescaling the time) has the normal form $\dot{x} = -x^3 + 0(x^4)$ on the center manifold; cf. Problem 6 in Section 2.12. And, as in Example 1, the universal unfolding is given by $\dot{x} = \mu_1 + \mu_2 x - x^3$, $\dot{y} = -y$. It follows that for $a = 0$ and $bd \neq 0$, this system has a codimension-two cusp bifurcation at $\boldsymbol{\mu} = \mathbf{0}$.

(c) For $a = b = 0$ and $cd \neq 0$, this system (after rescaling the time) has the normal form $\dot{x} = -x^4 + 0(x^5)$ on the center manifold; cf. Problem 6 in Section 2.12. And, as in equation (8), the universal unfolding is given by $\dot{x} = \mu_1 + \mu_2 x + \mu_3 x^2 - x^4$, $\dot{y} = -y$. It follows that for $a = b = 0$ and $cd \neq 0$, this system has a codimension-three swallow-tail bifurcation at $\boldsymbol{\mu} = \mathbf{0}$.

PROBLEM SET 4.4

1. (a) According to equation (3), $\sigma = 6\pi(a + b)$. Thus, σ has the same sign as $(a + b)$ and according to Theorem 1, if $(a + b) < 0$ then a unique stable limit cycle bifurcates from the origin as μ increases from zero; i.e., we have a supercritical Hopf bifurcation at the origin at the bifurcation value $\mu = 0$ if $(a + b) < 0$; and if $(a + b) > 0$ then a unique unstable limit cycle bifurcates from the origin as μ decreases from zero; i.e., we have a subcritical Hopf bifurcation at the origin at the bifurcation value $\mu = 0$ if $(a + b) > 0$.

(b) According to equation (3), $\sigma = 12\pi a$. Thus σ has the same sign as a and according to Theorem 1, if $a < 0$ then a unique stable limit cycle bifurcates from the origin as μ increases from zero, i.e., we have a supercritical Hopf bifurcation at the origin at the bifurcation value $\mu = 0$ if $a < 0$; and if $a > 0$ then a unique unstable limit cycle bifurcates from the origin as μ decreases from zero, i.e., we have a subcritical Hopf bifurcation at the origin at the bifurcation value $\mu = 0$ if $a > 0$. For this particular problem, we can also deduce that for $a < 0$ the origin is a stable, weak focus for $\mu = 0$ by computing $\dot{r} = ar^3$ for $\mu = 0$ and thus $\dot{r} < 0$ for $a < 0$ and $r > 0$. (Also, see p. 139 in the appendix.) We also note that, as in Definition 1 of Section 4.6, we have

$$\begin{vmatrix} P & Q \\ P_\mu & Q_\mu \end{vmatrix} = \begin{vmatrix} P & Q \\ x & y \end{vmatrix} = -r^2\left(1 + br^2\right) < 0$$

for the system in 1(b) provided $b \geq 0$ or provided $b < 0$ and $r^2 < 1/|b|$. Thus, the system in 1(b) defines a one-parameter family of negatively rotated vector fields with parameter μ according to Definition 1 in Section 4.6. A similar result holds for the system in 1(a).

2. (a) For example $f_{1y}(x, y) = -1 - x\cos\theta \to -1$ as $(x, y) \to \mathbf{0}$ since $|\cos\theta| < 1$ and $f_{1y}(0, 0) = \lim_{k\to 0}[f_1(0, k) - f_1(0, 0)]/k = \lim_{k\to 0}(-k/k) = -1$.

(b) $\dot{\theta} = 1$ and $\dot{r} = r(\mu - r)$; thus for $\mu > 0$ there is a unique stable limit cycle $r = \mu$ and for $\mu \le 0$, $\dot{r} < 0$ for $r \ne 0$ and there is no limit cycle around the origin. The phase portraits are similar to those in Figure 1.

(c) The bifurcation diagram is similar to that in Figure 2 except that $r = \mu$ is a cone in (x, y, μ)-space.

3. $\dot{x} = y$, $\dot{y} = -x - \mu y + x^3$. From equation (3′), $\sigma = 0$ since all of the coefficients a_{ij}, b_{ij} are zero except $b_{30} = 1$ and b_{30} does not appear in (3′). For $\mu = 0$ this is a Hamiltonian system with $H(x, y) = (y^2 + x^2)/2 - x^4/4$. The phase portrait is given in Figure 4 in Section 4.10.

5. The multiplicity of the weak focus at the origin is three. (See p. 139 in the appendix.)

6. Setting $(\rho - \mu^{1/3}/4)(\rho - \mu^{1/3})(\rho - 2\mu^{1/3})$ equal to $-a_1/2 - 3a_3\,\rho/8 - 5a_5\,\rho^2/16 - 35a_7/128\rho^3$ yields $a_1 = \mu$ (the coefficient of x in the problem), $a_3(\mu) = -22\mu^{2/3}/3$, $a_5(\mu) = 52\mu^{1/3}/5$ and $a_7 = -128/35$.

7. For $a_{20} = a$, $a_{11} = b$, $a_{02} = 0$, $b_{20} = \ell$, $b_{11} = m$ and $b_{02} = n$, equation (3) yields

$$\sigma = \frac{3\pi}{2}\,[-2a\ell + ab - m(n + \ell)] = \frac{3\pi}{2}\,W_1$$

with W_1 given in Theorem 4.

8. (a) $W_1 = -1 - m(n + 1)$. Thus, according to Theorem 4, this system with $\mu = 0$ has a weak focus of multiplicity one at the origin iff $m(n + 1) + 1 \ne 0$. If $m = 0$ or $n = -1$, $W_1 = -1$ (so $\sigma < 0$) and there is a supercritical Hopf bifurcation at the origin at the bifurcation value $\mu = 0$ according to Theorem 1.

(b) $W_2 = (2 + m)(3 - m)(n + 1)(n^2 - 2)$. Thus, according to Theorem 4, this system with $\mu = 0$ and $m(n + 1) + 1 = 0$ has a weak focus of multiplicity two at the origin iff $(m, n) \ne (-2, -1/2)$, $(3, -4/3)$ or $(-1/(1 \pm \sqrt{2}), \pm\sqrt{2})$. According to Theorem 5 and Table 1 in Section 4.6, for $W_2 > 0$ an unstable limit cycle bifurcates from the origin as μ decreases from 0 and for $W_2 < 0$ a stable limit cycle bifurcates from the origin as μ increases from zero.

(c) $W_3 = (2 + m)(2 + n)(n + 1)(n^2 - 2)$. Thus, according to Theorem 4, this system with $\mu = 0$ has a weak focus of multiplicity 3 iff $m(n + 1) + 1 = 0$ and $m = 3$, i.e., iff $m = 3$ and $n = -4/3$. In this case $W_3 > 0$ and, according to Theorem 5 in Section 4.6, an unstable limit cycle bifurcates from the origin as μ decreases from 0.

(d) For $(m, n) = (-2, -1/2)$, or $(-1/(1 \pm \sqrt{2}), \pm\sqrt{2})$, $W_1 = W_2 = W_3 = 0$ and according to Theorem 4, this system with $\mu = 0$ has a center. The system with $m = -2$ and $n = -1/2$ is Hamiltonian.

PROBLEM SET 4.5

1. Since $\nabla\cdot\mathbf{f}(\mathbf{x}, \mu) = 2\mu - 4|\mathbf{x}|^2$, $\nabla\cdot\mathbf{f}(\gamma_\mu(t), \mu) = -2\mu$ and $DP(r_\mu, \mu) = \exp[-4\mu\pi]$.

4. $\dot\theta = 1$, $\dot r = r[\mu - (r^2 - 1)^2][\mu - 4(r^2 - 1)^2]$ and $\dot r = 0$ for $r = \sqrt{1 \pm \mu^{1/2}}$ and $r = \sqrt{1 \pm \mu^{1/2}/2}$. It is then easy to show that $\gamma_1^\pm(t)$ and $\gamma_2^\pm(t)$ satisfy the given differential equations. $DP\left(\sqrt{1 \pm \mu^{1/2}}, \mu\right) = \exp\left[24\mu^{3/2}\left(\mu^{1/2} \pm 1\right)\right]\pi$ and $DP\left(\sqrt{1 \pm \mu^{1/2}/2}, \mu\right) = \exp\left[-4\mu^{3/2}\left(\mu^{1/2}/2 \pm 1\right)\right]\pi$.

5. Because of the xr and yr terms on the right-hand sides of the differential equations, this system is C^1 but not C^2; cf. Problem 2(a) in Section 4.4. In polar coordinates $\dot\theta = 1$ and $\dot r = r[\mu - (r - 1)^2]$ and $\dot r = 0$ for $r = 1 \pm \mu^{1/2}$, $\gamma^\pm(t) = (1 \pm \mu^{1/2})(\cos t, \sin t)^T$ defines two one-parameter families of periodic orbits. The bifurcations for this system are the same as those in Example 2 except that the stabilities of the periodic orbits and the critical point at the origin are reversed.

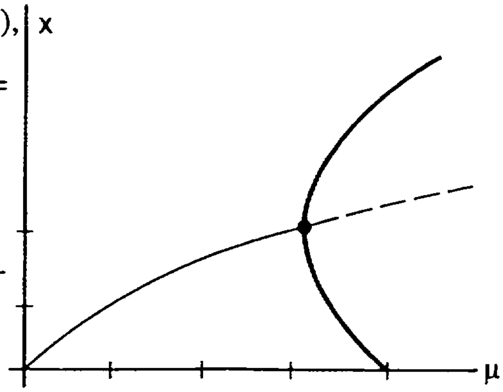
7. (a) $P(-1/2, -1/4) = -1/2$ and $DP(x, \mu) = -2x$ implies that $DP(-1/2, -1/4) = 1$; therefore, there is a nonhyperbolic fixed point $x = -1/2$ at the bifurcation value $\mu = -1/4$; the bifurcation diagram shows a saddle-node bifurcation at the point $(\mu, x) = (-1/4, -1/2)$.

(b) $P(0, 1) = 0$ and $DP(x, \mu) = \mu - 2\mu x$ implies that $DP(0, 1) = 1$; therefore, there is a nonhyperbolic fixed point $x = 0$ at the bifurcation value $\mu = 1$; the bifurcation diagram shows a transcritical bifurcation at the point $(\mu, x) = (1, 0)$.

8. (a) $P(-1/2, 0; -1/4) = (-1/2, 0)^T$ and $DP(\mathbf{x}, \mu) = [-2x, 0; 0, 2]$ implies that $DP(-1/2, 0; -1/4) = [-1, 0; 0, 2]$ which has an eigenvalue $\lambda_1 = -1$ of unit modulus; thus, here is a nonhyperbolic fixed point $(-1/2, 0)^T$ at the bifurcation value $\mu = -1/4$. The bifurcation diagram shows a saddle-node bifurcation at the point $(\mu, x, y) = (-1/4, -1/2, 0)$.

(b) $P(\mathbf{0}; 3/2) = \mathbf{0}$ and $DP(\mathbf{x}, \mu) = [0, 1; -1/2, \mu - 3y^2]$ implies that $DP(\mathbf{0}, 3/2) = [0, 1; -1/2, -3/2]$ which has an eigenvalue $\lambda = -1$ of unit modulus; thus, there is a nonhyperbolic fixed point $\mathbf{x} = \mathbf{0}$ at the bifurcation value $\mu = 3/2$. The bifurcation diagram shows a pitchfork bifurcation at the point $(\mu, x, y) = (3/2, 0, 0)$.

9. $DP(x, \mu) = -2x$ which implies that $DP\left(-1/2 + \sqrt{1+4\mu}/2, \mu\right) = 1 - \sqrt{1+4\mu}$ which is equal to -1 at $\mu = 3/4$; thus, $x = 1/2$ is a nonhyperbolic fixed point at the bifurcation value $\mu = 3/4$. For the map $F(x, \mu) = \mu - (\mu - x^2)^2$ we have $F(1/2, 3/4) = 1/2$, $DF(x, \mu) = 4x(\mu - x^2)$, $DF(1/2, 3/4) = 1$; $D^2F(x, \mu) = 4\mu - 12x^2$, $D^2F(1/2, 3/4) = 0$; $D^3F(x, \mu) = -36x$, $D^3F(1/2, 3/4) = -18 \neq 0$; $F_\mu(x, \mu) = -2(\mu - x^2)$, $F_\mu(1/2, 3/4) = 0$; $DF_\mu(x, \mu) = 4x$ and $DF_\mu(1/2, 3/4) = -2 \neq 0$. The equation $1 - (1 - x^2)^2 = x$ has the solutions $x = 0$, $x = 1$ and $x = \left(-1 \pm \sqrt{5}\right)/2$. The bifurcation diagram shows a period-doubling bifurcation at the point $(\mu, x) = (3/4, 1/2)$. Cf. Figure 7(a).



11. The one-dimensional maps in Problems 9 and 10 cannot be the Poincaré maps of any two-dimensional system of differential equations because period-doubling bifurcations cannot occur in two-dimensional systems since trajectories in the plane cannot cross.

PROBLEM SET 4.6

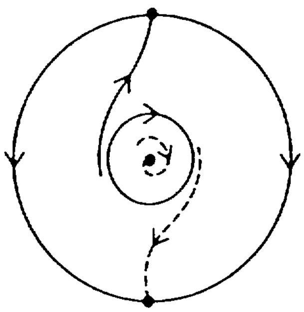
1. Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ with $\mathbf{f} = (P, Q)^T$ satisfies $\det [P, Q; P_\mu, Q_\mu] = PQ_\mu - QP_\mu > 0$ and that $\mathbf{y} = A(\mathbf{x})$ is a nonsingular transformation with $\det DA(\mathbf{x}) > 0$ (in a region $R \subset \mathbf{R}^2$). Let $DA = [a, b; c, d]$. Then the system $\dot{\mathbf{y}} = DA(\mathbf{x})\dot{\mathbf{x}} = DA(\mathbf{x})\mathbf{f}(\mathbf{x}, \mu) = (aP + bQ, cP + dQ)^T$ defines a one-parameter family of rotated vector fields if $\det [aP + bQ, cP + dQ; aP_\mu + bQ_\mu, cP_\mu + dQ_\mu] > 0$. But this determinant is equal to $\det [P, Q; P_\mu, Q_\mu]\,[a, c; b, d] = (PQ_\mu - QP_\mu)\cdot$
$\det DA(\mathbf{x})$ which is positive (in the region R).

2. (a) $\det [P, Q; P_\mu, Q_\mu] = r^2 > 0$ for $r > 0$; therefore, this system defines a one-parameter family of rotated vector fields. The conditions of Theorem 5, with $\mu_0 = 0$, are satisfied and since at $\mu = 0$ we have $\dot{r} = -r^2$, the origin is a stable weak focus. (It also follows from equation (2) in Section 4.4 that $\sigma = -12\pi$.) Thus, from Figure 1 with $\sigma < 0$ and $\omega < 0$ we have a supercritical Hopf bifurcation in which a stable limit cycle bifurcates from the origin as μ increases from 0. [This is quite clear since $\dot{r} = r(\mu - r)$.]

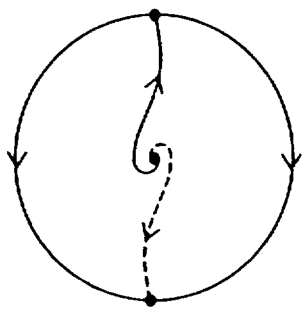
(b) As in Problem 2(b) in Section 4.4, the determinant of $[P, Q; P_\mu, Q_\mu] = -r^2(1 + br^2) < 0$ for $r^2 < 1/|b|$ (or for all r if $b \geq 0$) and this system defines a family of negatively rotated vector fields with parameter μ, according to Definition 1, in the neighborhood of the origin $0 \leq r < 1/\sqrt{|b|}$ for $b < 0$ (or for all $r \geq 0$ for $b \geq 0$). According to Theorem 5 and Table 1, for $\sigma = 9\pi a < 0$, a stable (positively oriented) limit cycle bifurcates from the origin as μ increases from 0 and for $\sigma > 0$ (i.e. for $a > 0$) an unstable (positively oriented) limit cycle bifurcates from the origin as μ decreases from 0. Thus for $a < 0$, the Hopf bifurcation is supercritical and for $a > 0$, the Hopf bifurcation is subcritical.

3. det $[P, Q; P_\mu, Q_\mu] = x^2(1 + x)^2 > 0$ except on the lines $x = 0$ and $x = -1$; therefore, this system forms a one-parameter family of rotated vector fields (mod $x(1 + x) = 0$), defined in the paragraph preceding Example 3. According to equation (3′) in Section 4.4, $\sigma = 9\pi/2 > 0$ for the system with $\mu = 0$; thus, the origin is an unstable weak focus and the flow swirls clockwise around $\mathbf{0}$, i.e., $\omega = -1$. It follows from Figure 1 (or from Theorem 1 in Section 4.4) that a unique unstable limit cycle bifurcates from the origin as μ decreases from the bifurcation value $\mu_0 = 0$; i.e., we have a subcritical Hopf bifurcation. Since $\mathbf{0}$ is the only critical point, Theorem 6 implies that the one-parameter family of limit cycles terminates as $\mu \to -\infty$ or as the limit cycle expands without bound.

4.

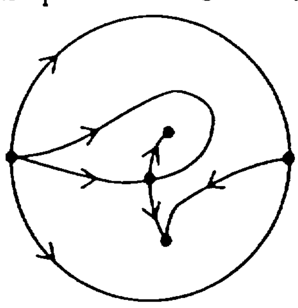


$-3 < \mu < 0$

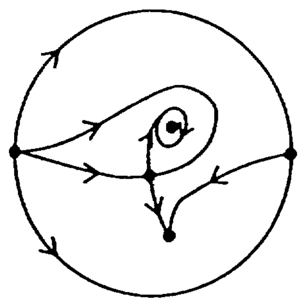


$0 \leq \mu < 1$

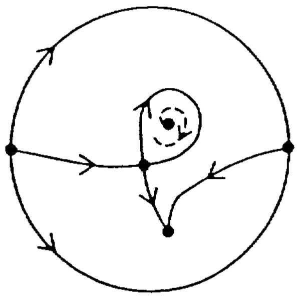
5. As in Example 3, it follows from the theory of rotated vector fields (cf. Theorems 1 and 6) that a unique, stable limit cycle is generated in a supercritical Hopf bifurcation at the critical point (1, 1) at the bifurcation value $\mu = 1/2$ and this stable, negatively oriented limit cycle expands monotonically as μ increases from the value $\mu = .5$ until, at $\mu = \mu_1 \cong .52$, it intersects the saddle at the origin and forms a separatrix cycle; cf. Figure 8. The global phase portraits are given by:



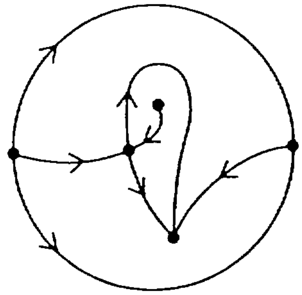
$-2 < \mu \leq 1/2$



$1/2 < \mu < \mu_1$



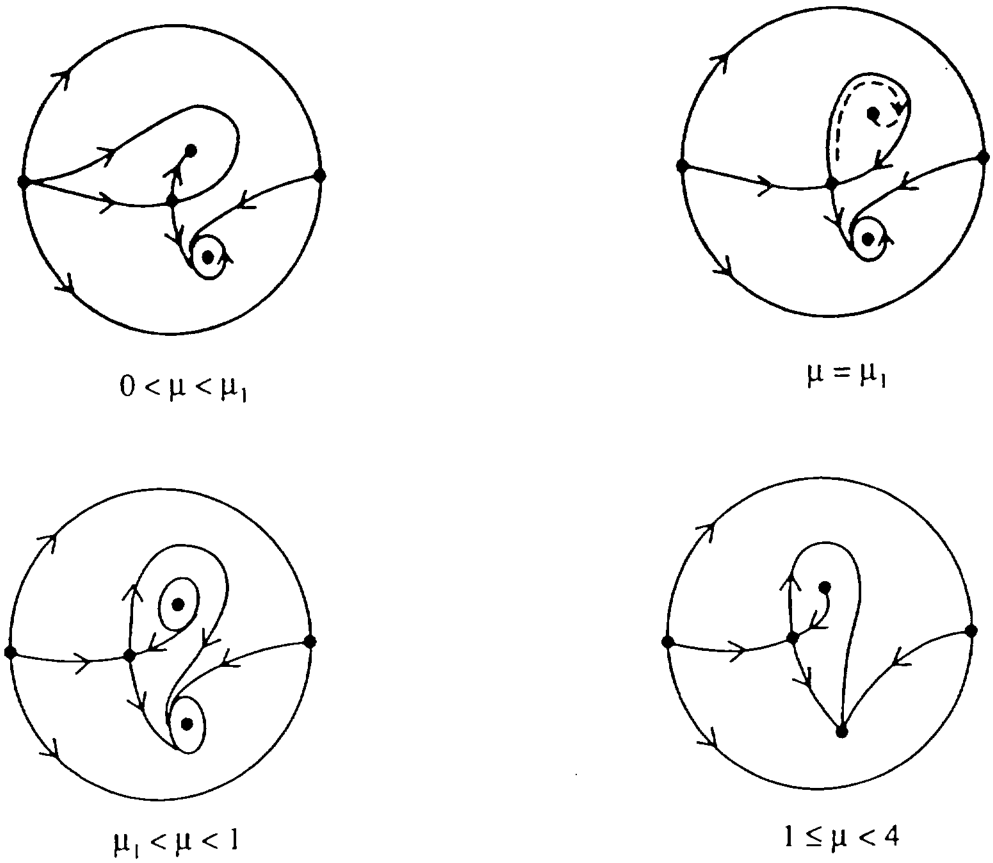
$\mu = \mu_1$



$\mu_1 < \mu < 2$

6. $[P, Q; P_\mu, Q_\mu] = y^2(1 + y)^2$. From equation (3′) in Section 4.4, $\sigma = -9\pi/2 < 0$ and clearly $\omega = -1$; therefore, by Figure 1 (or Theorem 1 in Section 4.4), there is a supercritical Hopf bifurcation in which a stable limit cycle bifurcates from the origin as the parameter μ increases from the bifurcation value $\mu = 0$. In order to draw the global phase portraits, you must determine the behavior of the critical points at infinity as in Section 3.10; e.g., for $\mu = 0$, there is a saddle node at the point $(\pm 1, 0, 0)$ at infinity and a critical point with two hyperbolic sectors at $(0, \pm 1, 0)$.

7. $[P, Q; P_\mu, Q_\mu] = (-x + y + y^2)^2 > 0$ for $x \neq y + y^2$. There are critical points at $(0, 0)$, $(2, 1)$ and $(12, -4)$; $Df(\mathbf{0}) = [-1, 1; -\mu, 4 + \mu]$ and thus $(0, 0)$ is a saddle. $Df(2, 1) = [-1, 3; -1 - \mu, 3\mu - 2]$, $\delta = 5$, $\tau = 3(\mu - 1)$ and thus $(2, 1)$ is a weak focus for $\mu = 1$; from (3′) in Section 4.4, with $a_{02} = 1$, $b_{11} = -1$ and $b_{02} = -1$ for $\mu = 1$, we find that $\sigma = 5\pi/2$ and clearly $\omega = -1$ at $(2, 1)$; thus, from Figure 1 (or Theorem 1 in Section 4.4) there is a subcritical bifurcation in which an unstable limit cycle bifurcates from $(2, 1)$ as μ decreases from $\mu_0 = 1$. $Df(12, -4) = [-1, -7; 4 - \mu, -7\mu + 8]$, $\delta = 20$, $\tau = 7 - 7\mu$ and thus $(12, -4)$ is a weak focus for $\mu = 1$; from (3′) in Section 4.4, with $a_{02} = 1$, $b_{11} = -1$ and $b_{02} = -1$ for $\mu = 1$, we find that $\sigma = -15\pi/56$ and clearly $\omega = +1$ at $(12, -4)$; thus from Figure 1 (or Theorem 1 in Section 4.4 with $a + d = 7 - 7\mu$ in place of μ as in Remark 1 in Section 4.4) there is a supercritical Hopf bifurcation in which a stable limit cycle bifurcates from $(12, -4)$ as μ decreases from $\mu_0 = 1$. For $0 < \mu < 4$, there is a saddle-node at $(\pm 1, 0, 0)$ on the equator of the Poincaré sphere and the global phase portraits are given by:



PROBLEM SET 4.7

1. For $\mu = \pm 1$, $Df(0) = [0, -1; 1, 0]$ and from equation (3) in Section 4.4, $\sigma = 15\pi/2$. Thus, by Theorem 1 in Section 4.4 a unique unstable limit cycle bifurcates from the origin as the trace $Df(0) = \mu^2 - 1$ decreases from zero, i.e., as μ decreases from $\mu_0 = 1$ or as μ increases from $\mu_0 = -1$. (Cf. Remark 1 in Section 4.4.)

2. Note that for $r = 0$ and $\mu = -1$, $\dot{r} = 0$ and we cannot determine the stability of the origin simply by computing $\dot{r}$; however, from equation (3) in Section 4.4, we find that for $\mu = -1$, $\sigma = -48\pi$ and then by Theorem 1 and Remark 1 in Section 4.4, we find that a unique stable limit cycle bifurcates from the origin in a supercritical Hopf bifurcation at the bifurcation value $\mu_0 = -1$ as the trace $Df(0, \mu) = (\mu - 1)^2 (\mu + 1)$ increases from zero, i.e., as μ increases from the value $\mu = -1$; cf. Figure 4. Note that for $\mu = 1$, equation (3) in Section 4.4 yields $\sigma = 0$; i.e., there is a weak focus of multiplicity two at the origin for $\mu = 1$; cf. Figure 4.

3. The bifurcation diagram is given by the graph of the relation $[(r^2-2)^2+\mu^2-1]\cdot[r^2+2\mu^2-2]=0$. (See p. 142 in the appendix.)

4. The bifurcation diagram is given by the graph of the relation $[(r^2-2)^2+\mu^2-1]\cdot[r^2+\mu^2-3]=0$. (See p. 143 in the appendix.)

PROBLEM SET 4.8

1. (a) Replacing (x, y) by (–x, –y) leaves the system invariant. $Df(\mathbf{0}) = [0, 1; 1, \mu]$ has $\delta = -1$ and therefore there is a saddle at the origin. (Cf. Sections 1.5 and 2.10.) $Df(\pm 1, 0) = [0, 2; -2, 1+\mu]$ has $\delta = 4$, $\tau = 1+\mu$ and there are foci at $(\pm 1, 0)$ for $|\mu| < 3$. (Cf. Sections 1.5 and 2.10.)

   (b) For $y \neq 0$, $[P, Q; P_\mu, Q_\mu] = y^2(1+r^2) > 0$.

   (c) After translating the origin to (1, 0) or (–1, 0) and using equation (3′) in Section 4.4, we see that $\sigma = -3\pi/4$ and since $\omega = -1$, it follows from Theorem 5 and Figure 1 in Section 4.6 that a unique, stable limit cycle bifurcates from each of the critical points (1, 0) and (–1, 0) as μ increases from the bifurcation value $\mu = -1$. According to Theorems 1 and 6 in Section 4.6, these limit cycles expand monotonically until they intersect the saddle at the origin simultaneously, in view of the symmetry, and form a grahic, S_0, with two homoclinic loops at $\mathbf{0}$ at the bifurcation value $\mu = \mu_1 > -1$.

   (d) By Theorem 3 in Section 4.6, since the limit cycles from part (c) are stable, the graphic S_0 is stable. Since $\sigma_0 \equiv \nabla\cdot f(\mathbf{0}) = \mu$, and since S_0 is stable, it follows that $\mu_1 \leq 0$. (Numerically deduce that $\mu_1 \cong -.74$.)

   (e) Since in Figure 1 of Section 4.6, the exterior stability of S_0, $\sigma = -1$, and since $\omega = -1$, a unique limit cycle bifurcates on the exterior of S_0 as μ increases from the bifurcation value $\mu_1 \cong -.74$ according to Theorem 3 in Section 4.6. This stable, negatively oriented limit cycle expands monotonically as μ increases. Since for large r, $\dot{r}$ is asymptotic to

$r^2 + x^2y^2/r$, there must be an unstable limit cycle which contracts with increasing μ and intersects the above-mentioned stable limit cycle at some bifurcation value $\mu = \mu_2 > \mu_1$ and forms a semistable limit cycle. The one-parameter family of limit cycles composed of these unstable and stable limit cycles thus extends from the graphic S_0 to infinity. Cf. the family L_ρ in Figure 9 of Section 4.9.

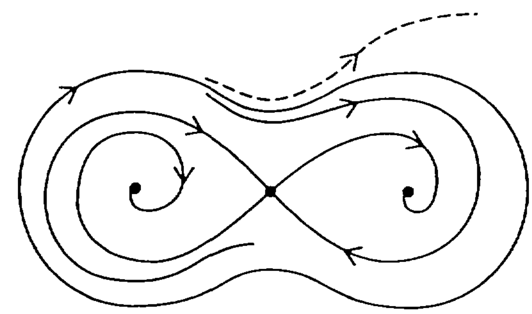
2. This problem is meant to illustrate how, using the theory of rotated vector fields, a separatrix cycle can be obtained from a Hamiltonian system and how limit cycles can then be made to bifurcate from the separatrix cycle by rotating the vector field.

(a) Clearly this is a Hamiltonian system with $H(x, y) = (y^2 - x^2)/2 + x^4/4$ and the phase portrait is given by Figure 3 in Section 4.9.

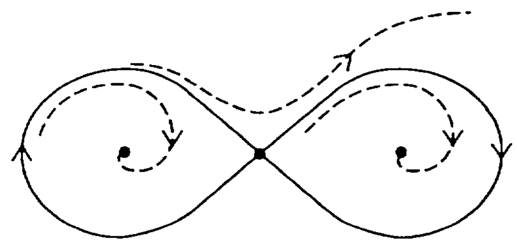
(b) The system in this part of the problem satisfies $[P, Q; P_\alpha, Q_\alpha] = H(x, y)\,[y^2 + (x - x^3)^2] < 0$ for $H(x, y) < 0$, i.e., inside the two loops of the graphic $H(x, y) = 0$. Therefore all vectors of the vector field inside these two loops rotate in the negative direction as α increases; thus, the two loops of the separatrix cycle, S_0, are internally unstable; i.e., according to the paragraph following Theorem 3 in Section 4.6, the negative of the interior stability $\sigma = -1$ and $\omega = -1$. Similarly on the exterior of S_0, we have a positively rotated vector field and therefore $\sigma = +1$ for the exterior stability of S_0.

(c) If we fix α at a positive value and embed the vector field of part (b) in a one-parameter family of rotated vector fields (5), then from Figure 1 and Theorem 3 in Section 4.6, an unstable limit cycle bifurcates on the interiors of the two loops of S_0 as μ increases from zero; and an unstable limit cycle bifurcates on the exterior of S_0 as μ decreases from zero. The trace of the linear part of (5) at $(\pm 1, 0)$ is given by $\tau_\mu = \text{trace } Df(\pm 1, 0, \mu) = -(3\alpha/4)\cdot \cos\mu + 3\sin\mu$ which is zero for $\mu = \mu^* \equiv \tan^{-1}(.1/4) \approx .025$ at $\alpha = .1$. The behavior of the system (5) at $\alpha = .1$ is described in the following phase portraits. Note that this system is invariant under $(x, y) \to (-x, -y)$.

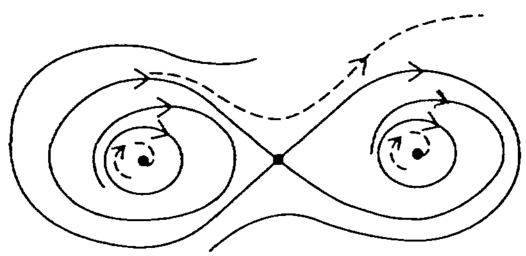
(d) For $\alpha = -.1$, the separatrix cycle is internally and externally stable and thus stable limit cycles bifurcate from the homoclinic loops on their interior as μ decreases from zero and on the exterior of S_0 as μ increases from zero.



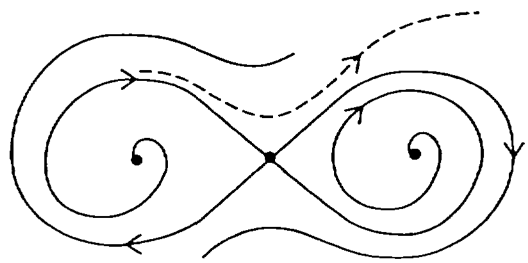
$\mu < 0$



$\mu = 0$



$0 < \mu < \mu^*$



$\mu \geq \mu^*$

4. (a) $\lambda_1 = 2$, $\lambda_2 = -1$, $\mathbf{v}_1 = (3, 1)^T$, $\mathbf{v}_2 = (0, 1)^T$, $E^s = \{\mathbf{0}\}$, $E^u = \text{Span}\ \{\mathbf{v}_1\}$, $E^c = \text{Span}\ \{\mathbf{v}_2\}$.

(b) $\lambda_1 = \lambda_2 = 1/2$, $\mathbf{v}_1 = (1, 0)^T$, gen. e. vect. $\mathbf{v}_2 = (0, 1)^T$, $E^s = \mathbf{R}^2$, $E^u = E^c = \{\mathbf{0}\}$.

(c) $\lambda_{1,2} = 1 \pm i$, $|\lambda_{1,2}| = \sqrt{2}$, $E^s = E^c = \{\mathbf{0}\}$, $E^u = \mathbf{R}^2$.

(d) $\lambda_1 = \left(3 + \sqrt{5}\right)/2$, $\lambda_2 = \left(3 - \sqrt{5}\right)/2$, $\mathbf{v}_1 = \left(2, 1 + \sqrt{5}\right)^T$, $\mathbf{v}_2 = \left(1 + \sqrt{5}, -2\right)^T$, $E^s = \text{Span}$ $\{\mathbf{v}_2\}$, $E^u = \text{Span}\ \{\mathbf{v}_1\}$, $E^c = \{\mathbf{0}\}$.

5. (a) $\mathbf{P}(x, y)$ and $\mathbf{P}^{-1}(x, y) = (x + y + x^3, x)$ are clearly continuously differentiable since they are polynomial maps.

(b) $(x, y) = (0, 0)$ is the only fixed point of $\mathbf{P}$; there is a one-dimensional stable manifold $W^s(\mathbf{0})$ tangent to $E^s = \text{Span}\ \left\{\left(1 + \sqrt{5}, 2\right)^T\right\}$ at the origin; and there is a one-dimensional unstable manifold $W^u(\mathbf{0})$ tangent to $E^u = \text{Span}\ \left\{\left(2, -1 - \sqrt{5}\right)^T\right\}$ at the origin.

6. If $x = .s_1s_2\cdots = \sum_{j=1}^{\infty} s_j/2^j$, then $F(x) = \sum_{j=1}^{\infty} s_j/2^{j-1}$ (mod 1) $= \sum_{j=1}^{\infty} s_{j+1}/2^j = .s_2s_3\cdots$; and $F^n(x) = \sum_{j=1}^{\infty} s_j/2^{j-n}$ (mod 1) $= \sum_{j=1}^{\infty} s_{j+n}/2^j = .s_{n+1}\,s_{n+2}\cdots$. It follows that $F(.\overline{0}) = .\overline{0}$ and $F(.\overline{1}) = .\overline{1}$; $F^2(.\overline{01}) = .\overline{01}$ and $F^2(.\overline{10}) = .\overline{10}$ etc. Thus, $.\overline{0L\ 01}$ (with n zeros) is a periodic orbit of period (n + 1). And if x and y differ in the nth place, then $x_n = 1$ and $y_n = 0$ or vice versa, so that $|F^{n-1}(x) - F^{n-1}(y)| = |.x_n\cdots - .y_n\cdots| = .1\cdots = 1/2 + \cdots \geq 1/2$.

PROBLEM SET 4.9

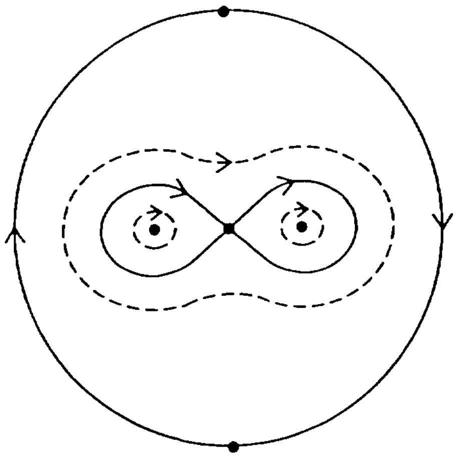
1. For $\varepsilon = 0$, $\gamma_\alpha(t) = (\alpha\cos t, \alpha\sin t)^T$ with $T_\alpha = 2\pi$. Thus, $M(\boldsymbol{\mu}, \alpha) =$

$$-\int_0^{2\pi} x_\alpha(t)\left[\mu_1 x_\alpha(t) + \mu_3 x_\alpha^3(t) + \mu_5 x_\alpha^5(t)\right]dt =$$

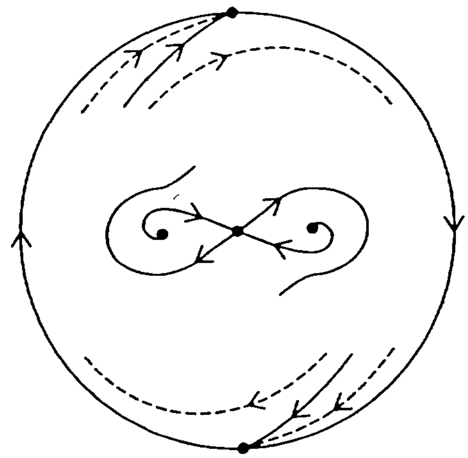
$$-\int_0^{2\pi}\left[\mu_1\alpha^2\cos^2 t + \mu_3\alpha^4\cos^4 t + \mu_5\alpha^6\cos^6 t\right]dt =$$

$-\pi\alpha^2\left[\mu_1 + 3\mu_3\alpha^2/4 + 5\mu_5\alpha^4/8\right]$; and $M(\boldsymbol{\mu}, \alpha) = 0$ iff $\alpha^2 =$ $\left[-3\mu_3 \pm \sqrt{9\mu_3^2 - 40\mu_1\mu_5}\right]/5\mu_5$. This yields two positive roots if $\mu_3\mu_5 < 0$ and $0 < \mu_1\mu_5 <$ $9\mu_3^2/40$. Thus, by Theorem 5, for all sufficiently small $\varepsilon \neq 0$, this Lienard system has exactly two limit cycles asymptotic to circles of radii r_1 and r_2, given by the square roots of the above two positive numbers, as $\varepsilon \to 0$.

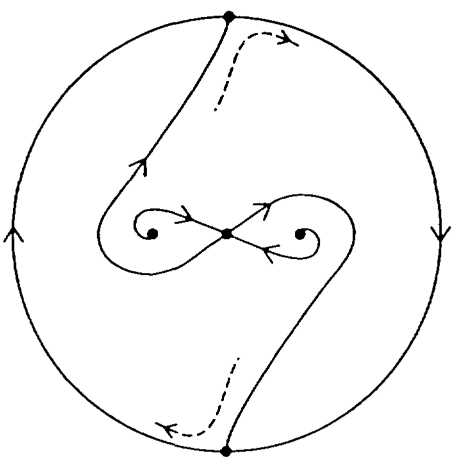
2. For $\alpha = 0$ (and $\beta = 0$) we have the phase portrait shown in Figure 3 and the global phase portrait shown below with critical points at $(0, \pm 1, 0)$ having two hyperbolic sectors. For $\varepsilon > 0$, the system (8) with $\beta = 0$ defines a one-parameter family of rotated vector fields (mod $y = 0$) with parameter α since $[P, Q; P_\alpha, Q_\alpha] = \varepsilon y^2 > 0$ for $y \neq 0$. Thus the field vectors rotate in the positive sense from those in Figure 3 as α increases from zero and we have the second phase portrait shown below. There will be a sequence of saddle-saddle bifurcation values α_n^*, described at the end of this section and in Figure 11, with $\alpha_n^* \to 0$. We thus have the following global phase portraits for (8) with $\beta = 0$ and $\varepsilon > 0$:



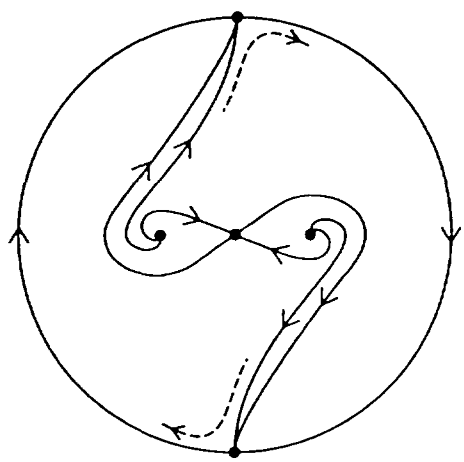
$\alpha = 0$



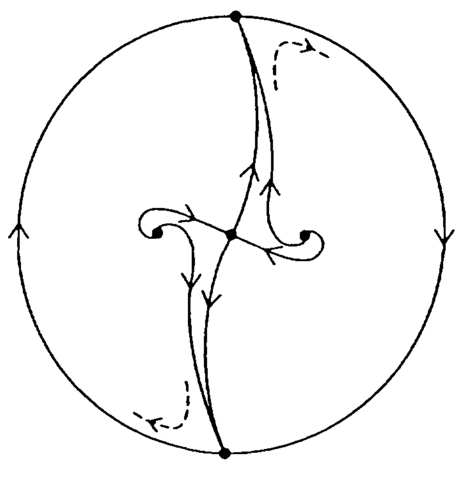
$\alpha > 0$



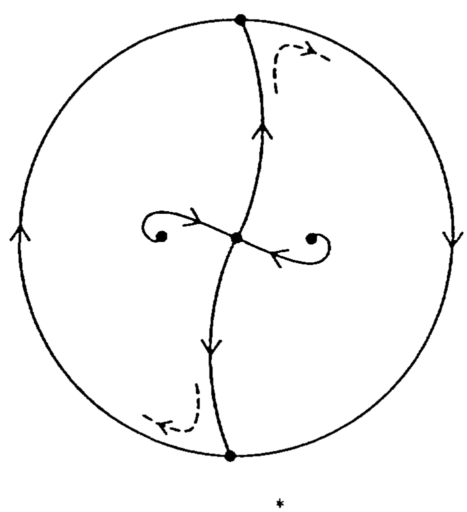
$\alpha = \alpha_1^*$



$\alpha_1^* < \alpha < \alpha_0^*$

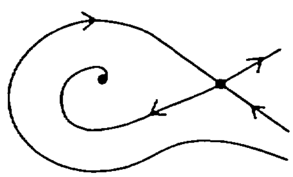


$\alpha > \alpha_0^*$

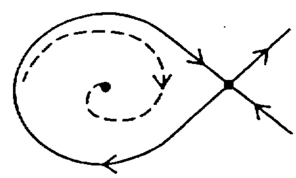


$\alpha = \alpha_0^*$

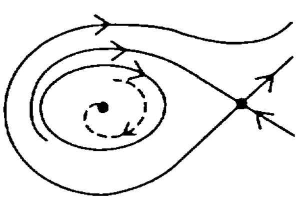
3. For $\varepsilon = 0$ the system is Hamiltonian with $H(x, y) = (y^2 - x^2)/2 - x^3/3$. The homoclinic loop through $(0, 0)$ is given by $H(x, y) = 0$ since $H(0, 0) = 0$. The homoclinic loop Γ_0 through the origin intersects the x-axis at $x = -3/2$. Using the symmetry and the fact that $dt = dx/y$, the Melnikov function along the homoclinic loop $y = \pm\sqrt{x^2 + 2x^3/3}$ is given by $M(\alpha) = \int_{-\infty}^{\infty} f(\gamma_0(t)) \wedge g(\gamma_0(t), \alpha)dt = \int_{-\infty}^{\infty} y_0^2(t)[\alpha + x_0(t)]dt =$
$2\int_{-3/2}^{0} \sqrt{x^2 + 2x^3/3}(\alpha + x)dx = 2\int_0^{-3/2} x\sqrt{1 + 2x/3}(\alpha + x)dx = 2\left(\frac{3}{5}\alpha - \frac{18}{35}\right)$; and $M(\alpha) = 0$ iff $\alpha = 6/7$. Thus, by Theorem 4, for sufficiently small $\varepsilon \neq 0$ there is an $\alpha(\varepsilon) = 6/7 + O(\varepsilon)$ such that this system has a homoclinic loop at the origin in a neighborhood of Γ_0. Since $Df(-1, 0) = [0, 1; -1, \varepsilon(\alpha - 1)]$, $\delta = 1$ and $\tau = \varepsilon(\alpha - 1) = 0$ at $\alpha = 1$. From (3′) in Section 4.4, $\sigma = +3\pi/2$ and clearly $\omega = -1$; thus, for $\varepsilon > 0$ there is a subcritical Hopf bifurcation at $(-1, 0)$ in which an unstable limit cycle bifurcates from $(-1, 0)$ as α decreases from the bifurcation value $\alpha = 1$; viz.



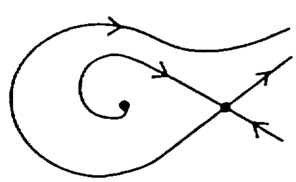
$\alpha < \alpha(\varepsilon)$



$\alpha = \alpha(\varepsilon) = 6/7 + O(\varepsilon)$



$\alpha(\varepsilon) < \alpha < 1$



$\alpha \geq 1$

4. (a) Integration by parts with $u = \cos\omega_0(t + t_0)$, $du = -\omega_0 \sin\omega_0(t + t_0)dt$, $dv = \operatorname{sech}t \tanh t\, dt$, $v = -\operatorname{sech}t$ yields $I = -\omega_0 \int_{-\infty}^{\infty} \operatorname{sech}t \cdot \sin\omega_0(t + t_0)dt = -\omega_0 \sin\omega_0 t_0 \int_{-\infty}^{\infty} \operatorname{sech}t \cdot \cos\omega_0 t\, dt$ since $\operatorname{sech}t \cdot \sin\omega_0 t$ is an odd function.

(b) Let C be the contour shown and $f(z) = \cos\omega_0 z/\cosh z$. Then $\cosh z$ has a simple zero at $z = i\pi/2$ and $f(z)$ has a simple pole at $z = i\pi/2$ and no other singularities inside or on C. Thus, the residue of $f(z) = g(z)/h(z)$ at $z = i\pi/2$, $\operatorname{Res} f(z) = g(i\pi/2)\big/h'(i\pi/2) = \cos(i\omega_0\pi/2)\big/\sinh(i\pi/2) = \cosh(\omega_0\pi/2)\big/i = -i\cosh(\omega_0\pi/2)$.

(c) For $z = \pm a + i\tau$ and $0 \le \tau \le \pi$, $\left|\cos\omega_0 z/\cosh z\right| \le e^{-a}\left(1 + e^{\omega_0\pi}\right) / \left(1 - e^{-2a}\right)$ and therefore, $\left|\int_0^{\pi} \frac{\cos\omega_0(\pm a + i\tau)}{\cosh(\pm a + i\tau)} d\tau\right| \le \pi e^{-a}\left(1 + e^{\omega_0\pi}\right) / \left(1 - e^{-2a}\right) \to 0$ as $a \to \infty$. And for $z = t + i\pi = -u + i\pi$, $\int_{a+i\pi}^{a-i\pi} f(z)dz = \int_{-a}^{a} \frac{\cos\omega_0(-u + i\pi)}{\cosh(-u + i\pi)} du = \cosh\omega_0\pi \int_{-a}^{a} \frac{\cos\omega_0 u}{\cosh u} du$ since $\cosh(-u + i\pi) = -\cosh u$, $\cos\omega_0(-u + i\pi) = \cos\omega_0 u \cosh\omega_0\pi + \sin\omega_0 u \cdot \sinh\omega_0\pi$ and $\sin\omega_0 u/\cosh u$ is an odd function.

(d) We then have $\oint_C f(z)dz = 2\pi i \operatorname{Res} f(z) = \left(1 + \cosh\omega_0\pi\right)\int_{-a}^{a} \frac{\cos\omega_0 t}{\cosh t} dt + \int_0^{\pi} \frac{\cos\omega_0(a + i\tau)}{\cosh(a + i\tau)} d\tau + \int_{\pi}^{0} \frac{\cos\omega_0(-a + i\tau)}{\cosh(-a + i\tau)} d\tau$. And, finally, letting $a \to \infty$ and using the results from (a)–(c), we have $I = \dfrac{-\omega_0 \sin\omega_0 t_0 2\pi\cosh(\omega_0\pi/2)}{\left(1 + \cosh\omega_0\pi\right)} = -\pi\omega_0 \sin\omega_0 t_0$ $\operatorname{sech}(\omega_0\pi/2)$ since $2\cosh^2(\omega_0\pi/2)/(1 + \cosh\omega_0\pi) = 1$.

PROBLEM SET 4.10

1. The given differential equation is equivalent to the second-order differential equation $\ddot{x} = \dot{y} = x - x^3 + \varepsilon(\alpha\dot{x} + \beta x^2\dot{x}) = x - x^3 + \varepsilon\dot{x}g(x, \boldsymbol{\mu})$ with $g(x, \boldsymbol{\mu}) = \alpha + \beta x^2$. Similarly, the Lienard equation is equivalent to $\ddot{x} = \dot{y} + \varepsilon(\mu_1\dot{x} + 3\mu_2 x^2\dot{x}) = x - x^3 + \varepsilon(\mu_1 + 3\mu_2 x^2)\dot{x}$ which is equivalent to the above second-order differential equation with $\alpha = \mu_1$ and $\beta = 3\mu_2$.

2. For $\lambda_i = \varepsilon\lambda_{i1} + \varepsilon^2\lambda_{i2} + \cdots$ and $\gamma_0(t) = (x_0(t), y_0(t)) = (\alpha\cos t, \alpha\sin t)$, the (first-order) Melnikov function for this system is given by

$$\begin{aligned} M(\alpha, \boldsymbol{\mu}) &= -\int_0^{2\pi}\Big[\lambda_{11}y_\alpha^2(t) + \lambda_{21}x_\alpha^2(t)y_\alpha(t) + (2\lambda_{31} + \lambda_{41})x_\alpha(t)y_\alpha^2(t) \\ &\qquad -\lambda_{21}y_\alpha^3(t) + \lambda_{11}x_\alpha^2(t) - \lambda_{31}x_\alpha^3(t) + (2\lambda_{21} + \lambda_{51})x_\alpha^2(t)y_\alpha(t) + \lambda_{61}y_\alpha^2(t)x_\alpha(t)\Big]dt \\ &= -\lambda_{11}\int_0^{2\pi}\left[x_\alpha^2(t) + y_\alpha^2(t)\right]dt = -2\pi\alpha^2\lambda_{11}; \end{aligned}$$

and we have $M(\alpha, \boldsymbol{\mu}) \equiv 0$ for all $\alpha > 0$ iff $\lambda_{11} = 0$.

3. The system in Example 3 is equivalent to $\ddot{x} = \dot{y} + \varepsilon(\mu_1\dot{x} + 3\mu_2 x^2\dot{x}) = -x + x^3 + \varepsilon(\mu_1 + 3\mu_2 x^2)\dot{x}$ and similarly, the system given in this problem is equivalent to $\ddot{x} = \dot{y} = -x + x^3 + \varepsilon\dot{x}(\mu_1 + 3\mu_2 x^2)$. For $\varepsilon = 0$ the given system is clearly Hamiltonian with $H(x, y) = (x^2 + y^2)/2 - x^4/4$ and the two heteroclinic orbits are given by $2(x^2 + y^2) - x^4 = 1$ since $H(\pm 1, 0) = 1/4$. Along the upper heteroclinic orbit $y = (1 - x^2)/\sqrt{2}$ and the Melnikov function along the upper (or lower) homoclinic orbit is given by

$$\begin{aligned} M(\boldsymbol{\mu}) &= \int_{-\infty}^{\infty} \mathbf{f}(\gamma_0(t)) \wedge \mathbf{g}(\gamma_0(t), \boldsymbol{\mu}) = \int_{-\infty}^{\infty} y_0^2(t)\left[\mu_1 + 3\mu_2 x_0^2(t)\right]dt \\ &= \int_{-1}^{1} y\left(\mu_1 + 3\mu_2 x^2\right)dx = 1/\sqrt{2}\int_{-1}^{1}\left(1 - x^2\right)\left(\mu_1 + 3\mu_2 x^2\right)dx = 2\sqrt{2}\left(\mu_1/3 + \mu_2/5\right). \end{aligned}$$

4. The function $x_\alpha(t)$ given in Example 2 satisfies $\dot{x}_\alpha(t) = \dfrac{\sqrt{2}}{2-\alpha^2}\,\mathrm{dn}'(u)$ and $\ddot{x}_\alpha(t) = \dfrac{\sqrt{2}}{\left(2-\alpha^2\right)^{3/2}}\,\mathrm{dn}''(u)$ with $u = t/\sqrt{2-\alpha^2}$. Thus, $\ddot{x}_\alpha - x_\alpha + x_\alpha^3 =$ $\sqrt{2}/\left(2-\alpha^2\right)^{3/2}\left[\mathrm{dn}''(u) - \left(2-\alpha^2\right)\mathrm{dn}(u) + 2\mathrm{dn}^3(u)\right] \equiv 0$. Also $\dot{x}_\alpha(t) =$ $\sqrt{2}/\left(2-\alpha^2\right)\cdot \mathrm{dn}'(u) = -\sqrt{2\alpha^2}/\left(2-\alpha^2\right)\cdot \mathrm{sn}(u)\cdot\mathrm{cn}(u) = y_\alpha(t)$ given in Example 2.

5. The function $x_\alpha(t)$ given in Example 3 satisfies $\dot{x}_\alpha(t) = \dfrac{\sqrt{2}\alpha}{1+\alpha^2}\,\mathrm{sn}'(u)$ and $\ddot{x}_\alpha(t) =$ $\dfrac{\sqrt{2}\alpha}{\left(1+\alpha^2\right)^{3/2}}\,\mathrm{sn}''(u)$ with $u = t/\sqrt{1+\alpha^2}$. Thus, $\ddot{x}_\alpha + x_\alpha - x_\alpha^3 = \sqrt{2}\alpha/\left(1+\alpha^2\right)^{3/2}\cdot$ $\left[\mathrm{sn}''(u) + \left(1+\alpha^2\right)\mathrm{sn}(u) - 2\alpha^2\mathrm{sn}^3(u)\right] \equiv 0$. Also $\dot{x}_\alpha(t) = \sqrt{2}\alpha/\left(1+\alpha^2\right)\cdot\mathrm{sn}'(u) =$ $\sqrt{2}\alpha/\left(1+\alpha^2\right)\cdot\mathrm{cn}(u)\cdot\mathrm{dn}(u) = y_\alpha(t)$ given in Example 3.

6. For the system in Example 2 along the exterior periodic orbit $\gamma_\alpha(t)$ given in this problem, with $T_\alpha = 4K(\alpha)\sqrt{2\alpha^2-1}$ and $u = t/\sqrt{2\alpha^2-1}$,

$$M(\alpha,\boldsymbol{\mu}) = \int_0^{T_\alpha}\left[\mu_2 x_\alpha^6(t) + (\mu_1-\mu_2)x_\alpha^4(t) - \mu_1 x_\alpha^2(t)\right]dt$$

$$= \int_0^{4K(\alpha)}\left[\mu_2\left(\frac{2\alpha^2}{2\alpha^2-1}\right)^3\mathrm{cn}^6(u) + (\mu_1-\mu_2)\left(\frac{2\alpha^2}{2\alpha^2-1}\right)^2\mathrm{cn}^4(u) - \mu_1\frac{2\alpha^2}{\left(2\alpha^2-1\right)}\mathrm{cn}^2(u)\right]\sqrt{2\alpha^2-1}\,du$$

$$= \frac{2\alpha^2}{\left(2\alpha^2-1\right)^{5/2}}\int_0^{4K(\alpha)}\left[\mu_2 4\alpha^4\mathrm{cn}^6(u) + (\mu_1-\mu_2)2\alpha^2\left(2\alpha^2-1\right)\mathrm{cn}^4(u) - \mu_1\left(2\alpha^2-1\right)^2\mathrm{cn}^2(u)\right]du$$

$$= \frac{8}{\left(2\alpha^2-1\right)^{5/2}}\left\{\frac{4\mu_2}{15}\left[\left(23\alpha^4-23\alpha^2+8\right)E(\alpha) + \left(1-\alpha^2\right)\left(15\alpha^4-19\alpha^2+8\right)K(\alpha)\right]\right.$$

$$+\frac{2(\mu_1-\mu_2)\left(2\alpha^2-1\right)}{3}\left[2\left(2\alpha^2-1\right)E(\alpha) + \left(2-3\alpha^2\right)\left(1-\alpha^2\right)K(\alpha)\right]$$

$$\left.-\mu_1\left(2\alpha^2-1\right)^2\left[E(\alpha) - \left(1-\alpha^2\right)K(\alpha)\right]\right\}$$

$$= \frac{8}{\left(2\alpha^2-1\right)^{5/2}}\left\{\frac{\mu_1}{3}\left[\left(2\alpha^2-1\right)^2 E(\alpha)-\left(2\alpha^2-1\right)\left(\alpha^2-1\right)K(\alpha)\right]\right.$$

$$\left.+\frac{6}{15}\mu_2\left[2\left(\alpha^4-\alpha^2+1\right)E(\alpha)-\left(\alpha^4-3\alpha^2+2\right)K(\alpha)\right]\right\}.$$

And therefore we have $M(\alpha, \boldsymbol{\mu}) = 0$ if μ_1/μ_2 is equal to the function of α given in this problem, whose graph is shown in Figure 6. Note that the value $\mu_1/\mu_2 = -2.4$ for $\alpha = 1$ corresponds to the homoclinic loop bifurcation as in Example 2.

7. For the system in Problem 7 with $\gamma_\alpha(t) = (x_\alpha(t), y_\alpha(t))$ the interior periodic orbit given in Example 2, with $T_\alpha = 2K(\alpha)\sqrt{2-\alpha^2}$ and $u = t/\sqrt{2-\alpha^2}$,

$$M(\alpha, \boldsymbol{\mu}) = \int_0^{T_\alpha}\left[\mu_2 x_\alpha^5(t)+\mu_1 x_\alpha^4(t)-\mu_2 x_\alpha^3(t)-\mu_1 x_\alpha^2(t)\right]dt$$

$$= \frac{1}{2}\int_0^{4K(\alpha)}\left[\mu_2\frac{4\sqrt{2}}{\left(2-\alpha^2\right)^{5/2}}\mathrm{dn}^5(u)+\mu_1\frac{4}{\left(2-\alpha^2\right)^2}\mathrm{dn}^4(u)-\mu_2\frac{2\sqrt{2}}{\left(2-\alpha^2\right)^{3/2}}\mathrm{dn}^3(u)\right.$$

$$\left.-\mu_1\frac{2}{\left(2-\alpha^2\right)}\mathrm{dn}^2(u)\right]\sqrt{2-\alpha^2}\,du$$

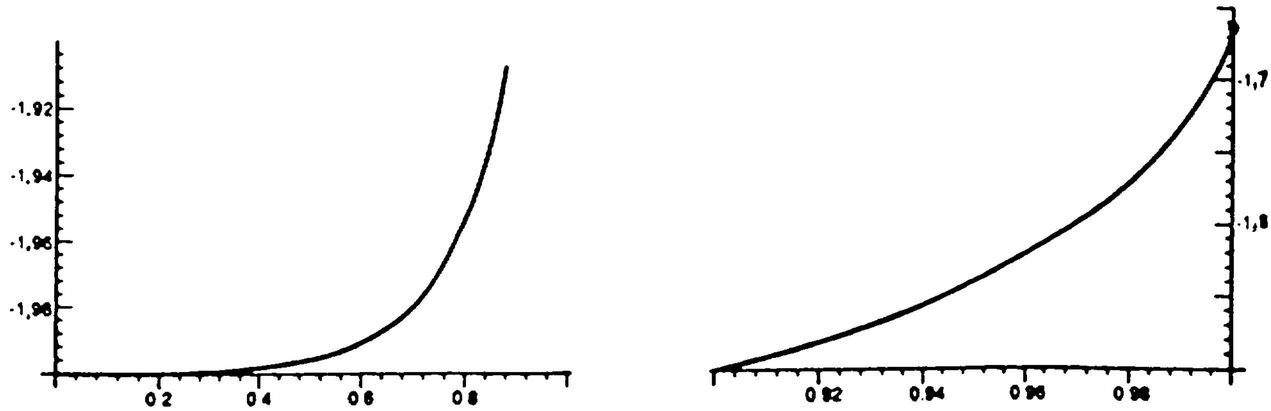
$$= \frac{\sqrt{2}\mu_2}{\left(2-\alpha^2\right)^2}\left[\frac{\pi}{2}\left(8-8\alpha^2+3\alpha^4\right)-\left(2-\alpha^2\right)^2\pi\right]$$

$$+\frac{\mu_1}{\left(2-\alpha^2\right)^{3/2}}\left\{\frac{8}{3}\left[\left(\alpha^2-1\right)K(\alpha)+2\left(2-\alpha^2\right)E(\alpha)\right]-4\left(2-\alpha^2\right)E(\alpha)\right\}$$

$$= \frac{1}{\left(2-\alpha^2\right)^{3/2}}\left\{\frac{\mu_2\sqrt{2}\pi\alpha^4}{2\left(2-\alpha^2\right)^{1/2}}+\frac{4\mu_1}{3}\left[\left(2-\alpha^2\right)E(\alpha)+2\left(\alpha^2-1\right)K(\alpha)\right]\right\}$$

and therefore $M(\alpha, \boldsymbol{\mu}) = 0$ if $\mu_1/\mu_2 = 3\pi\alpha^4/4\sqrt{2}\left(2-\alpha^2\right)^{1/2}\left[\left(\alpha^2-2\right)E(\alpha)+2\left(1-\alpha^2\right)K(\alpha)\right]$.

The graph of the ratio μ_1/μ_2 as a function of α is shown below:

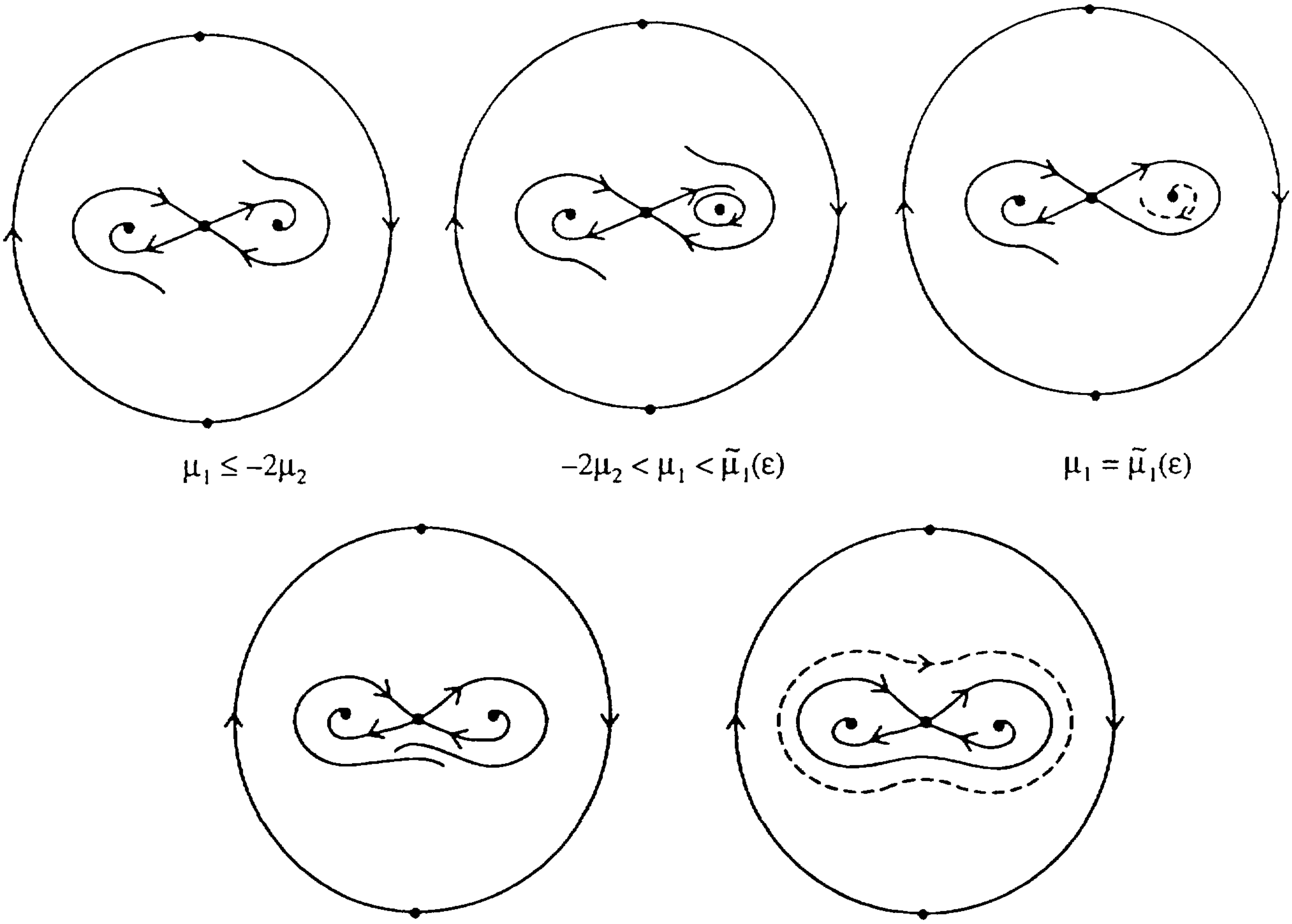


And we see that there is a unique limit cycle for $-2\mu_2 < \mu_1 < \tilde{\mu}_1(\varepsilon) \cong -1.67\mu_2$. It is shown in Problem 8 that $\tilde{\mu}_1(\varepsilon) = -3\pi\mu_2/4\sqrt{2} + 0(\varepsilon)$ is the homoclinic loop bifurcation value and, using equation (3′) in Section 4.4, we can show that this system has a supercritical Hopf bifurcation at $\mu_1 = -2\mu_2$.

8. The system in Problem 7 is equivalent to the second-order differential equation $\ddot{x} = \dot{y} + \varepsilon(\mu_1\dot{x} + 2\mu_2 x\dot{x}) = x - x^3 + \varepsilon\dot{x}(\mu_1 + 2\mu_2 x)$ as is the system in Problem 8. The Melnikov function along the homoclinic loop $\gamma_0^+(t)$ which, as in Example 3 in Section 4.9, lies on the curve $y^2 = x^2 - x^4/2$, is given by

$$\begin{aligned} M(\boldsymbol{\mu}) &= \int_{-\infty}^{\infty} \mathbf{f}(\gamma_0^+(t)) \wedge \mathbf{g}(\gamma_0^+, \boldsymbol{\mu})dt \\ &= \int_{-\infty}^{\infty} y_0^2(t)[\mu_1 + 2\mu_2 x_0(t)]dt = 2\int_0^{\sqrt{2}} x\sqrt{1 - x^2/2}\,(\mu_1 + 2\mu_2 x)dx \\ &= 2\left(2\mu_1/3 + \sqrt{2}\pi\mu_2/4\right). \end{aligned}$$

And therefore, $M(\boldsymbol{\mu}) = 0$ if $\mu_1 = -3\sqrt{2}\,\pi\,\mu_2/8$ which leads to the homoclinic loop bifurcation value $\tilde{\mu}_1(\varepsilon) = -3\sqrt{2}\,\pi\,\mu_2/8 + 0(\varepsilon)$ according to Theorem 3. Note that for $\mu_1 = 0$ the system in Problem 7 is symmetric about the y-axis and it therefore has a continuous band of cycles for $\mu_1 = 0$ (even if $\varepsilon \neq 0$). The phase portraits for the asymmetric perturbed Duffing oscillator in Problem 7 with $\varepsilon > 0$, $\mu_2 > 0$ and $\mu_1 \leq 0$ are shown below:



$\mu_1 \le -2\mu_2$ $\quad$ $-2\mu_2 < \mu_1 < \tilde{\mu}_1(\varepsilon)$ $\quad$ $\mu_1 = \tilde{\mu}_1(\varepsilon)$

$\tilde{\mu}_1(\varepsilon) < \mu_1 < 0$ $\quad$ $\mu_1 = 0$

9. For $\gamma_\alpha(t) = (x_\alpha(t), y_\alpha(t)) = (\alpha\cos t, \alpha\sin t)$ and $T_\alpha = 2\pi$, the Melnikov function is given by

$$\begin{aligned} M(\alpha, \boldsymbol{\mu}) &= -\int_0^{2\pi}\left[\mu_1 x_\alpha^2(t) + \mu_3 x_\alpha^4(t) + \mu_5 x_\alpha^6(t) + \mu_7 x_\alpha^8(t)\right]dt \\ &= -2\pi\alpha^2\left[\frac{\mu_1}{2} + \frac{3\mu_3}{8}\alpha^2 + \frac{5\mu_5}{16}\alpha^4 + \frac{35\mu_7}{128}\alpha^6\right] \end{aligned}$$

which clearly has three positive roots for appropriate choices of $\boldsymbol{\mu}$; in fact, if we want three particular sized limit cycles, say limit cycles asymptotic to circles of radii $r = 1, 2, 3$ as $\varepsilon \to 0$, we simply set the polynomial $(\alpha^2 - 1)(\alpha^2 - 4)(\alpha^2 - 9)$, i.e., $\alpha^6 - 14\alpha^4 + 49\alpha^2 - 36$ equal to the above 6th degree polynomial in α in order to determine that $\mu_1 = -72$, $\mu_3 = 392/3$, $\mu_5 = -224/5$ and $\mu_7 = 128/35$ will produce three limit cycles of the desired sizes. (Cf. Theorem 6 in Section 3.8.)

PROBLEM SET 4.11

1. The solution of the unperturbed problem through a point $x = \alpha$ on the x-axis at time $t = 0$ is $\gamma_\alpha(t) = (\alpha\cos t, -\alpha\sin t)$. In the context of Theorem 1, we have $M_1(\alpha, \lambda) \equiv 0$, $H(x, y) = y^2/2 - U(x)$ with $U(x) = -x^2/2$, $f(x, y, \varepsilon) = \varepsilon\lambda_{12}x - 2x^2 - (2\lambda_{21} + \lambda_{51})xy + y^2$, $g(x, y, \varepsilon) = \varepsilon\lambda_{12}y - \lambda_{21}x^2 + (4 + \lambda_{41})xy + \lambda_{21}y^2$, $F(x, y) = \int_0^y f(x, s, 0)ds - \int_0^x g(s, 0, 0)ds = -2x^2y - (\lambda_{21} + \lambda_{51}/2)xy^2 + y^3/3 + \lambda_{21}x^3/3$, $G(x, y) = g(x, y, 0) + F_x(x, y) = \lambda_{41}xy - \lambda_{51}y^2/2$, $G_1(x, y) = \lambda_{41}xy$, $G_2(x, y) = -\lambda_{51}y^2/2$, $G_{1h}(x, y) = \lambda_{41}x/y$, $P_2(x, h) = \int_0^x \lambda_{51}(s^2 - 2h)/2\, ds = \lambda_{51}x^3/6 - \lambda_{51}hx$ and $P_{2h}(x, h) = -\lambda_{51}x$. Thus, according to Theorem 1, with $dx = ydt$ and $dy = -xdt$ to zero order and with $h = \alpha^2/2$, we have, with the integrals taken around $\Gamma_\alpha : \mathbf{x} = \gamma_\alpha(t)$,

$$
\begin{aligned}
M_2(\alpha, \lambda) &= \oint[G_{1h}P_2 - G_1P_{2h}]dx + \oint[g_\varepsilon dx - f_\varepsilon dy] - \oint\frac{F}{y}[f_x + g_y]dx \\
&= \oint\left[\lambda_{41}\frac{x}{y}\left(\frac{\lambda_{51}}{6}x^3 - \lambda_{51}hx\right) - \lambda_{41}xy(-\lambda_{51}x)\right]dx + \oint[\lambda_{12}ydx - \lambda_{12}xdy] \\
&\quad - \oint\left[\frac{-2x^2y - (\lambda_{21} + \lambda_{51}/2)xy^2 + y^3/3 + \lambda_{21}x^3/3}{y}\right]\cdot \\
&\qquad\qquad [-4x - (2\lambda_{21} + \lambda_{51})y + (4 + \lambda_{41})x + 2\lambda_{21}y]dx \\
&= \lambda_{41}\lambda_{51}\oint\left[\frac{x^4}{6y} - \frac{hx^2}{y} + x^2y\right]dx + \lambda_{12}\oint[ydx - xdy] \\
&\quad - \oint\left[2\lambda_{51}x^2y^2 - \frac{\lambda_{51}}{3}y^4 - \lambda_{41}(\lambda_{21} + \lambda_{51}/2)x^2y^2 + \frac{\lambda_{41}\lambda_{21}}{3}x^4\right]\frac{dx}{y} \\
&= \lambda_{41}\lambda_{51}\int_0^{2\pi}\left[\frac{\alpha^4}{6}\cos^4 t - h\alpha^2\cos^2 t + \alpha^4\cos^2 + \sin^2 t\right]dt + \lambda_{12}\alpha^4\int_0^{2\pi}[\sin^2 t + \cos^2 t]dt \\
&\quad - \alpha^4\int_0^{2\pi}\left[2\lambda_{51}\cos^2 t\sin^2 t - \frac{\lambda_{51}}{3}\sin^4 t - \lambda_{41}(\lambda_{21} + \lambda_{51}/2)\cos^2 t\sin^2 t + \frac{\lambda_{41}\lambda_{21}}{3}\cos^4 t\right]dt \\
&= \lambda_{41}\lambda_{51}\alpha^4\left[\frac{1}{6}\cdot\frac{3}{4}\pi - \frac{\pi}{2} + \frac{\pi}{4}\right] + \lambda_{12}\alpha^2 2\pi -
\end{aligned}
$$

$$- \quad \alpha^4\left[2\lambda_{51}\cdot\frac{\pi}{4}-\frac{\lambda_{51}}{3}\cdot\frac{3\pi}{4}-\lambda_{41}\left(\lambda_{21}+\lambda_{51}/2\right)\frac{\pi}{4}+\frac{\lambda_{41}\lambda_{21}}{3}\cdot\frac{3\pi}{4}\right]$$

$$= \quad -\frac{\pi}{8}\lambda_{41}\lambda_{51}\alpha^4+\frac{\alpha^4\pi}{4}\left(-\lambda_{51}+\frac{\lambda_{41}\lambda_{51}}{2}\right)+\lambda_{12}\alpha^2\cdot 2\pi$$

$$= \quad \pi\alpha^2\left(2\lambda_{12}-\frac{\alpha^2}{4}\lambda_{51}\right)=4\pi h\left(\lambda_{12}-h\lambda_{51}/4\right)$$

which agrees with the formula for $d_2(\alpha, \lambda) = M_2(\alpha, \lambda)/\alpha$ in Lemma 1. (Cf. equation (3) in Section 4.10 with $\omega_0 = -1$ and $\left|f(\gamma_\alpha(0))\right| = \alpha$.)

2. Letting $y \rightarrow -y$ we get the system in the form of equation (1_μ): $\dot{x} = y + \varepsilon(\varepsilon ax + y^2 - 8xy - 2x^2)$, $\dot{y} = -x + \varepsilon(\varepsilon ay + 4xy)$. We therefore have $M_1(\alpha, a) \equiv 0$, $H(x, y) = y^2/2 - U(x)$ with $U(x) = -x^2/2$, $f(x, y, \varepsilon) = \varepsilon ax + y^2 - 8xy - 2x^2$, $g(x, y, \varepsilon) = \varepsilon ay + 4xy$, $F(x, y) = y^3/3 - 4xy^2 - 2x^2y$, $G(x, y) = -4y^2$, $G_1(x, y) = 0$, $G_2(x, y) = -4y^2$, $P_2(x, h) = 2xh - x^3/3$, and $P_{2h}(x, h) = 2x$. Thus, from Theorem 1 with $dx = ydt$ and $dy = -xdt$ to zero order and with $h = \alpha^2/2$, we have, with the integrals being taken around $\gamma_\alpha(t) = (x_\alpha(t), y_\alpha(t)) = (\alpha\cos t, -\alpha\sin t)$,

$$\begin{aligned}
M_2(\alpha, a) &= \oint\left[G_{1h}P_2 - G_1P_{2h}\right]dx + \oint\left[g_\varepsilon dx - f_\varepsilon dy\right] - \oint\frac{F}{y}\left[f_x + g_y\right]dx \\
&= a\oint\left[ydx - xdy\right] - \oint\left[y^3/3 - 4xy - 2x^2\right](-8y)dx \\
&= a\int_0^{2\pi}\left[y_\alpha^2(t) + x_\alpha^2(t)\right]dt + 8\int_0^{2\pi}\left[y_\alpha^4(t)/3 - 4x_\alpha(t)y_\alpha^3(t) - 2x_\alpha^2(t)y_\alpha^2(t)\right]dt \\
&= a2\pi\alpha^2 + 8\left(\frac{\alpha^4}{3}\cdot\frac{3\pi}{4} - 2\alpha^2\frac{\pi}{4}\right) = -2\pi\alpha^2\left(a - \alpha^2\right).
\end{aligned}$$

And since, from equation (3) in Section 4.10, $d_2(\alpha, a) = -M_2(\alpha, a)/\alpha$ (in view of the fact that $\omega_0 = +1$ and $x = \alpha$), we have $d_2(\alpha, a) = 2\pi\alpha(a - \alpha^2)$ or $d(\alpha, \varepsilon, a) = 2\pi\varepsilon^2\alpha(a - \alpha^2) + O(\varepsilon^3)$ which agrees with the formula for $d(\alpha, \varepsilon, a)$ following Corollary 1 to $O(\varepsilon^3)$. Using Lemma 1, we can show that the error is $O(\varepsilon^6)$ as in that formula; cf. Problem 3.

3. In order to use Lemma 1 for the system in Corollary 1, we must note that $\lambda_{11} = \lambda_{21} = 0$, $\lambda_{31} = 2$, $\lambda_{41} = 0$, $\lambda_{51} = 8$, $\lambda_{61} = 0$, $\lambda_{12} = a$ and all of the other $\lambda_{ij} = 0$. Thus, from Lemma 1, we have $d_1(x, \boldsymbol{\lambda}) = 2\pi\lambda_{11}x = 0$, $d_2(x, \boldsymbol{\lambda}) = 2\pi\lambda_{12}x - \pi\lambda_{51}x^3/4 = 2\pi ax - \pi 8x^3/4 = 2\pi x(a - x^2)$ and $d_3(x, \boldsymbol{\lambda}) = d_4(x, \boldsymbol{\lambda}) = d_5(x, \boldsymbol{\lambda}) = 0$. Since $\lambda_{41} \neq -5$, we can not use the formula for $d_6(x, \boldsymbol{\lambda})$ and there may well be $0(\varepsilon^6)$ terms in $d(x, \varepsilon, \boldsymbol{\lambda})$. Thus $d(x, \varepsilon, \boldsymbol{\lambda}) = \varepsilon d_1(x, \boldsymbol{\lambda}) + \varepsilon^2 d_2(x, \boldsymbol{\lambda}) + \cdots = 2\pi\varepsilon^2 x(a - x^2) + 0(\varepsilon^6)$ as in the formula for $d(x, \varepsilon, \boldsymbol{\lambda})$ following Corollary 1. It then follows from Theorem 2 that for $a > 0$ and all sufficiently small $\varepsilon \neq 0$, the system (4) has exactly one hyperbolic limit cycle in an $0(\varepsilon)$ neighborhood of the circle of radius $x = \sqrt{a}$. This completes the proof of Corollary 1. Corollaries 2 and 3 are proved in a similar fashion by using Lemma 1 to derive the given formulas for the displacement function $d(x, \varepsilon, \boldsymbol{\mu})$ and by using Theorem 2 and the results cited in Remark 1.

4. First of all, it is easy to see that for $\boldsymbol{\gamma}_\alpha(t) = (x_\alpha(t), y_\alpha(t)) = (\alpha\cos t, -\alpha\sin t)$, $M_1(\alpha) = \int_0^{2\pi}\left[cx_\alpha^4(t)y_\alpha(t) + bx_\alpha^2(t)y_\alpha(t) + ex_\alpha^5(t)\right]dt \equiv 0$ for all α. And then with $U(x) = -x^2/2$, $f(x, y, \varepsilon) = \varepsilon ax + bxy + \varepsilon dx^3 + ex^4$, $g(x, y, \varepsilon) = cx^4$, $f_\varepsilon(x, y, \varepsilon) = ax + dx^3$, $g_\varepsilon = 0$, $f_x(x, y, 0) = by + 4ex^3$, $g_y = 0$, $F(x, y) = bxy^2/2 + ex^4y - cx^5/5$, $G(x, y) = by^2/2 + 4ex^3y$, $G_1(x, y) = 4ex^3y$, $G_2(x, y) = by^2/2$, $G_{1h}(x, y) = 4ex^3/y$, $P_2(x, h) = b(hx - x^3/b)$, $P_{2h}(x, h) = bx$. Thus, from Theorem 1 with $dx = ydt$ and $dy = -xdt$ to zero order and with $h = \alpha^2/2$, we have, with the integrals being taken around $\boldsymbol{\gamma}_\alpha(t)$,

$$
\begin{aligned}
M_2(\alpha) &= \oint\left[G_{1h}P_2 - G_1P_{2h}\right]dx + \oint\left[g_\varepsilon dx - f_\varepsilon dy\right] - \oint\frac{F}{y}\left[f_x + g_y\right]dx \\
&= \oint\left[\frac{4ex^3}{y}b\left(hx - \frac{x^3}{6}\right) - 4ex^3ybx\right]dx - \oint\left(ax + dx^3\right)dx \\
&\quad - \oint\left(bxy^2/2 + ex^4y - cx^5/5\right)\left(by + 4ex^3\right)\frac{dx}{y} \\
&= 4be\int_0^{2\pi}\left[hx_\alpha^4(t) - x_\alpha^6(t)/6 - x_\alpha^4(t)y_\alpha^2(t)\right]dt + \int_0^{2\pi}\left[ax_\alpha^2(t) + dx_\alpha^4(t)\right]dt -
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^{2\pi}\left[3bcx_\alpha^4(t)y_\alpha^2(t) - 4cx_\alpha^8(t)/5\right]dt \\
&= \pi\alpha^2\left[\frac{7}{16}c\alpha^6 + \frac{5}{24}bc\alpha^4 + \frac{3}{4}d\alpha^2 + a\right]
\end{aligned}
$$

which follows using the formulas for $\int_0^{2\pi}\cos^{2m} tdt$ given at the end of Section 4.10.

5. For $\gamma_\alpha(t) = (-\alpha\cos t, \alpha\sin t)$ we have $M_1(\alpha) \equiv 0$. And then with $f(x, y, \varepsilon) = \varepsilon ax + \varepsilon bx^3 + \varepsilon cx^5 + x^6$, $g(x, y, \varepsilon) = Ax^6 + Bx^4 + Cx^2$, $f_x(x, y, 0) = 6x^5$, $f_\varepsilon(x, y, \varepsilon) = ax + bx^3 + cx^5$, $g_y = 0$, $g_\varepsilon = 0$, $F(x, y) = x^6y - Ax^7/7 - Bx^5/5 - Cx^3/3$, $G(x, y) = 6x^5y$, $G_1(x, y) = 6x^5y$, $G_2(x, y) = 0$, $G_{1h}(x, y) = 6x^5/y$, $P_2(x, h) = 0$, $P_{2h}(x, h) = 0$. Thus, from Theorem 1 with $dx = ydt$, $dy = -xdt$ to zero order and with $h = \alpha^2/2$ and the integrals being taken around $\gamma_\alpha(t)$, we have

$$
\begin{aligned}
M_2(\alpha) &= \oint[G_{1h}P_2 - G_1P_{2h}]dx + \oint[g_\varepsilon dx - f_\varepsilon dy] - \oint\frac{F}{y}[f_x + g_y]dx \\
&= -\oint\left[ax + bx^3 + cx^5\right]dy - \oint\left[x^6y - Ax^7/7 - Bx^5/5 - Cx^3/3\right]\left[6x^5/y\right]dx \\
&= \int_0^{2\pi}\left[ax_\alpha^2(t) + bx_\alpha^4(t) + cx_\alpha^6(t)\right]dt + \int_0^{2\pi}\left[6Ax_\alpha^{12}(t)/7 + 6Bx_\alpha^{10}(t)/5 + 2Cx_\alpha^8(t)\right]dt \\
&= \pi\alpha^2\left(\frac{99}{256}A\alpha^{10} + \frac{189}{320}B\alpha^8 + \frac{35}{32}C\alpha^6 + \frac{5}{8}c\alpha^4 + \frac{3}{4}b\alpha^2 + a\right)
\end{aligned}
$$

where we use the integrals of even powers of cost given at the end of Section 4.10 and the formulas $\int_0^{2\pi}\cos^{10} tdt = \frac{63\pi}{2^7}$ and $\int_0^{2\pi}\cos^{12} tdt = \frac{3\cdot 77\pi}{2^9}$ which follow from the formula $\frac{1}{2\pi}\int_0^{2\pi}\cos^{2m} tdt = \binom{2m}{m}\cdot\frac{1}{2^{2m}}$ given in Theorem 6 in Section 3.8.

7. With $\gamma_\alpha(t) = (x_\alpha(t), y_\alpha(t)) = (\alpha\cos t, -\alpha\sin t)$, we have $M_1(\alpha) =$

$$\int_0^{2\pi}\left[a + bx_\alpha(t) + x_\alpha(t)y_\alpha(t) + cx_\alpha^3(t) + 3x_\alpha^2(t)y_\alpha(t) - y_\alpha^3(t)\right]y_\alpha(t)dt = 3\pi/4 - 3\pi/4 \equiv 0.$$

And then $f(x, y, \varepsilon) = 0$, $g(x, y, \varepsilon) = a + bx + xy + cx^3 + 3x^2y - y^3$, $g_y(x, y, \varepsilon) =$ $x + 3x^2 - 3y^2$, $g_\varepsilon = 0$, $F(x, y) = -ax - bx^2/2 - cx^4/4$, $G(x, y) = xy + 3x^2y - y^3$, $G_1(x, y) = xy + 3x^2y - y^3$, $G_2 = P_2 = P_{2h} = 0$ and $G_{1h}(x, y) = (x + 3x^2 - 3y^2)/y$ imply that

$$\begin{aligned} M_2(\alpha) &= -\oint \frac{F}{y}\left[g_y\right]dx = \oint\left[ax + bx^2/2 + cx^4/4\right]\left[x + 3x^2 - 3y^2\right]\frac{dx}{y} \\ &= \int_0^{2\pi}\left[ax_\alpha^2(t) + 3bx_\alpha^4(t)/2 - 3bx_\alpha^2(t)y_\alpha^2(t)/2 + 3cx_\alpha^6(t)/4 - 3cx_\alpha^4(t)y_\alpha^2(t)/4\right]dt \\ &= \pi\alpha^2\left[a + 3b\alpha^2/4 + 3c\alpha^4/8\right] = \pi h\left[2a + 3bh + 3ch^2\right] \end{aligned}$$

where we have used the fact that $h = \alpha^2/2$, $\int_0^{2\pi}\cos^2 t\sin^2 t\,dt = \pi/4$, $\int_0^{2\pi}\cos^4 t\sin^2 t\,dt = \pi/8$ and the formulas for the integrals of even powers of cost given at the end of Section 4.10.

PROBLEM SET 4.12

1. (a) You should find $Q_{40} = a_{30} - b_{21} + a_{12}$, $Q_{31} = b_{30}$, $Q_{22} = a_{12}/2$, $Q_{13} = (b_{12} - a_{21} + 3b_{30} + a_{03})/4$, $Q_{04} = b_{03}/4$, $Q_{30} = (a_{20} - b_{11} + 2a_{02})/3$, $Q_{21} = b_{20}$, $Q_{12} = a_{02}$, $Q_{03} = (b_{02} + 2b_{20} - a_{11})/3$, $Q_{20} = a_{10}/2$, $Q_{11} = b_{10}$, $Q_{02} = b_{01}/2$, $Q_{10} = a_{00}$ and $Q_{01} = b_{00}$; $q(x, y) = (b_{21} - a_{12})x^2 - (3b_{12} - a_{12} + 3b_{30} + a_{03})xy/4 + (b_{11} - 2a_{02})x + (a_{11} - 2b_{20})y$; and $\alpha(h) = (a_{21} - b_{12} + 3a_{03} - 3b_{30})h + (a_{01} - b_{10})$.

(b) $Q_{50} = (3a_{04} - 3b_{31} + 2a_{22} - 2b_{13} + 8a_{04})/15$, $Q_{41} = b_{40}$, $Q_{32} = (a_{22} - b_{13} + 4a_{04})/3$, $Q_{23} = (b_{22} - a_{31} + 4b_{40})/3$, $Q_{14} = a_{04}$ and $Q_{05} = (3b_{04} - 3a_{13} + 2b_{22} - 2a_{31} + 8b_{40})/15$; $q_{30} = b_{31} - 2(a_{22} - b_{13} + 4a_{04})/3$, $q_{21} = a_{31} - 4b_{40}$, $q_{12} = b_{13} - 4a_{04}$, and $q_{03} = a_{13} - 2(b_{22} - a_{31} + 4b_{04})/3$; and
40
$\alpha(h)$ is the same as in part (a).

(c) This problem is done in Iliev's paper [58].

2. (a) For the system in Problem 2(a), we have $f_1(x, y) = -2x^2 - (2\lambda_{21} + \lambda_{51})xy + y^2$, $g_1(x, y) = -\lambda_{21}x^2 + (4 + \lambda_{41})xy + \lambda_{21}y^2$, $f_2(x, y) = \lambda_{12}x$ and $g_2(x, y) = \lambda_{12}y$. Therefore $\Omega_1 = \omega_1 = g_1dx - f_1dy = [-\lambda_{21}x^2 + (4 + \lambda_{41})xy + \lambda_{21}y^2]dx + [2x^2 + (2\lambda_{21} + \lambda_{51})xy - y^2]dy$, and $\omega_2 = g_2dx - f_2dy = \lambda_{12}ydx - \lambda_{12}xdy$. It then follows from the proof of Lemma 1 with n = 2 that $q_1(x, y) = (b_{11} - 2a_{02})x + (a_{11} - 2b_{20})y = (2\lambda_{21} + \lambda_{51} - 2\lambda_{21})x + (4 + \lambda_{41} - 4)y = \lambda_{51}x + \lambda_{41}y$. And finally, $d_2(h) = \int_{H=h}\Omega_2 = \int_{H=h}(\omega_2 + q_1\omega_1) = \lambda_{12}\int_0^{2\pi}(y^2 + x^2)dt + \int_0^{2\pi}(\lambda_{51}x + \lambda_{41}y)$ $[-\lambda_{21}x^2y + (4 + \lambda_{41})xy^2 + \lambda_{21}y^3 - 2x^3 - (2\lambda_{21} + \lambda_{51})x^2y + xy^2]dt = 2\pi\alpha^2\lambda_{12} - \int_0^{2\pi}[2\lambda_{51}x^4 - \lambda_{51}x^2y^2 + \lambda_{41}(2\lambda_{21} + \lambda_{51})x^2y^2 - \lambda_{51}(4 + \lambda_{41})x^2y^2 + \lambda_{21}\lambda_{41}x^2y^2 - \lambda_{21}\lambda_{41}y^4]dt = 2\pi(\alpha^2\lambda_{12} - \alpha^4\lambda_{51}/8) = 4\pi h\lambda_{12} - \pi h^2\lambda_{51}$ where we have used $dx = ydt$, $dy = -xdt$, $x_\alpha(t) = \alpha\cos t$, $y_\alpha(t) = \alpha\sin t$, $\int_0^{2\pi}\sin^2t\cos^2t\,dt = 2\pi/8$, $\int_0^{2\pi}\sin^4t\,dt = \int_0^{2\pi}\cos^4t\,dt = 2\pi\cdot 3/8$, $\int_0^{2\pi}\sin t\cos^3t\,dt = \int_0^{2\pi}\cos t\sin^3t\,dt = 0$ and $\alpha = \sqrt{2h}$.

(b) For $\lambda_{11} = \lambda_{12} = \lambda_{51} = 0$, consider the system in Problem 2(b): We have $f_1(x, y) = -2x^2 - 2\lambda_{21}xy + y^2$, $g_1(x, y) = -\lambda_{21}x^2 + (4 + \lambda_{41})xy + \lambda_{21}y^2$, $f_2(x, y) = -\lambda_{52}xy$, $g_2(x, y) = 0$, $f_3(x, y) = \lambda_{13}x$ and $g_3(x, y) = \lambda_{13}y$. Then $q_1(x, y) = \lambda_{41}y$ (from part a with $\lambda_{51} = 0$), $\Omega_2 = \omega_2 + q_1\omega_1 = \lambda_{52}xydy + \lambda_{41}y\left\{[-\lambda_{21}x^2 + (4 + \lambda_{41})xy + \lambda_{21}y^2]dx + [2x^2 + 2\lambda_{21}xy - y^2]dy\right\} = \lambda_{52}xydy + [-\lambda_{41}\lambda_{21}x^2y + \lambda_{41}(4 + \lambda_{41})xy^2 + \lambda_{21}\lambda_{41}y^3]dx + [2\lambda_{41}x^2y + 2\lambda_{41}\lambda_{21}xy^2 -$

$\lambda_{41}y^3]dy$. Thus, in determining $q_2(x, y)$, we have $a_{30} = 0$, $a_{21} = -\lambda_{41}\lambda_{21}$, $a_{12} = \lambda_{41}(4 + \lambda_{41})$, $a_{03} = \lambda_{21}\lambda_{41}$, $b_{30} = 0$, $b_{21} = 2\lambda_{41}$, $b_{12} = 2\lambda_{41}\lambda_{21}$, $b_{03} = -\lambda_{41}$ and $b_{11} = \lambda_{52}$. And from Problem 1(a), we have $q_{02} = 0$ and $q_2(x, y) = b_{11}x + (b_{21} - a_{12})x^2 - (3b_{12} - a_{12} + 3b_{30} + a_{03})xy/4 = \lambda_{52}x - (2\lambda_{41} + \lambda_{41}^2)x^2 - (7\lambda_{21}\lambda_{41} - 4\lambda_{41} - \lambda_{41}^2)xy/4$. Therefore, $\int_{H=h}\Omega_3 = \int_{H=h}\omega_3 + q_1\omega_2 + q_2\omega_1 = \lambda_{13}\int_0^{2\pi}(y^2 + x^2)dt - \lambda_{41}\lambda_{52}\int_0^{2\pi}x^2y^2dt + \int_0^{2\pi}[\lambda_{52}(4 + \lambda_{41})x^2y^2 - \lambda_{52}2x^4 + \lambda_{52}x^2y^2 + 0(x^3y, xy^3, x^5, x^4y, x^3y^2, x^2y^3, xy^4)]dt = 2\pi\alpha^2\lambda_{13} - \pi\alpha^4\lambda_{52}/4 = \pi h[4\lambda_{13} - \lambda_{52}h]$ where we have used $dx = ydt$, $dy = -xdt$, the integrals in part (a), and $\int_0^{2\pi}\sin^mt\cos^nt\,dt = 0$ for m + n odd.

3. For $n = 1$ and $\Omega = (a_{10}x + a_{01}y + a_{00})dx + (b_{10}x + b_{01}y + b_{00})dy$, we have $\Omega = dQ + qdH + \alpha(H)ydx + \beta(H)xydx$ for $Q(x, y) = Q_{20}x^2 + Q_{11}xy + Q_{02}y^2 + Q_{10}x + Q_{01}y + B_{00}H(x, y)$, $q(x, y) = q_{00}$, $\alpha(h) = \alpha_0$, $\beta(h) = 0$ and $H(x, y) = (x^2 + y^2)/2 - x^3/3$ provided $\Omega = \left[2Q_{20}x + Q_{11}y + Q_{10} + B_{00}(x - x^2) + q_{00}(x - x^2) + \alpha_0 y\right]dx + \left[Q_{11}x + 2Q_{02}y + Q_{01} + B_{00}y + q_{00}y\right]dy$; i.e., provided $2Q_{20} + B_{00} + q_{00} = a_{10}$, $Q_{11} + \alpha_0 = a_{01}$, $Q_{10} = a_{00}$, $B_{00} + q_{00} = 0$, $Q_{11} = b_{10}$, $2Q_{02} + B_{00} + q_{00} = b_{01}$ and $Q_{01} = b_{00}$. We can choose $B_{00} + q_{00} = 0$ (here and for all $n \geq 1$), $Q_{20} = a_{10}/2$, $Q_{10} = a_{00}$, $Q_{11} = b_{10}$, $Q_{02} = b_{01}/2$, $Q_{01} = b_{00}$, and $\alpha_0 = a_{01} - b_{10}$. For $n = 2$, you should find, in addition to the above formulas, that $Q_{30} = b_{11}/30 - a_{02}/15 + a_{20}/3$, $Q_{21} = b_{20}$, $Q_{12} = a_{02}/5 + 3b_{11}/10$, $Q_{03} = b_{20}/3$, $B_{10} = -3q_{10}/4$, $q_{10} = 4b_{11}/5 - 8a_{02}/5$ and $B_{01} = q_{01} = 0$.

PROBLEM SET 4.13

1. As in Section 2.12, we let $y = h(x) = ax^2 + bx^3 + \cdots$ be the Taylor series for the analytic center manifold. Substituting this expansion into equation (5) in Section 2.12 results in $(2ax + 3bx^2)(ax^2 + bx^3) - \mu_2(ax^2 + bx^3) - (x^2 + ax^3) + 0(x^4) \equiv 0$, i.e., $(-\mu_2 a - 1)x^2 + 0(x^3) \equiv 0$ which implies that $a = -1/\mu_2$ or that the analytic center manifold is approximated by $y = -x^2/\mu_2 + 0(x^3)$ as $x \to 0$. Equation (6) in Section 2.12 then implies that the flow on the analytic center manifold is approximated by $\dot{x} = -x^2/\mu_2 + 0(x^3)$ as $x \to 0$. And for either $\mu_2 > 0$ or $\mu_2 < 0$, this implies that there is a saddle-node at the origin with two hyperbolic sectors in the right-half plane for $\mu_2 \neq 0$.

2. For $\mu_2 \neq 0$, the linear part of the system (2) has a simple eigenvalue $\lambda = 0$ with corresponding eigenvector $\mathbf{v} = (1, 0)^T$ and $[Df(\mathbf{0}, 0, \mu_2)]^T = [0, 0; 1, \mu_2]$ has an eigenvector $\mathbf{w} = (\mu_2, -1)^T$ corresponding to the eigenvalue $\lambda = 0$. Furthermore, $\mathbf{w}^T\mathbf{f}_{\mu_1}(\mathbf{0}, 0, \mu_2) = (\mu_2, -1)\,(0, 1)^T = -1 \neq 0$ and $\mathbf{w}^T[D^2\mathbf{f}(\mathbf{0}, 0, \mu_2)\,(\mathbf{v}, \mathbf{v})] = (\mu_2, -1)\,(0, 2)^T = -2 \neq 0$. Thus, for $\mu_2 \neq 0$, according to Theorem 1 in Section 4.2, the system (2) experiences a saddle-node bifurcation at the equilibrium point $\mathbf{x} = \mathbf{0}$ as the parameter μ_1 passes through the bifurcation value $\mu_1 = 0$.

3. Setting $\nu_1 = -1/4$ and translating the origin to the center of the system (4) at $(-1/2, 0)$, i.e., letting $x = u + 1/2$ and $y = v$, we obtain $\dot{x} = y$ and $\dot{y} = -x + x^2 + \varepsilon(\alpha y + \beta xy)$ with $\alpha = \nu_2 - 1/2$ and $\beta = 1$. This system is equivalent to $\ddot{x} = \dot{y} = -x + x^2 + \varepsilon(\alpha\dot{x} + \beta x\,\dot{x})$ as is the system $\dot{x} = y + \varepsilon(ax + bx^2)$, $\dot{y} = -x + x^2$ with $(a, b) = (\alpha, \beta/2)$. The Melnikov function along the one-parameter family of periodic orbits $\gamma_\alpha(t) = (x_\alpha(t), y_\alpha(t))$ of period $T_\alpha = 8(1 - \alpha^2 + \alpha^4)^{1/4}\,K(\alpha)$, given in this problem is given by

$$
\begin{aligned}
M(\alpha, \boldsymbol{\mu}) &= \int_0^{T_\alpha} \mathbf{f}(\gamma_\alpha(t)) \wedge \mathbf{g}(\gamma_\alpha(t), \boldsymbol{\mu})dt = \int_0^{T_\alpha}\left[ax_\alpha^2(t) + (b-a)x_\alpha^3(t) - bx_\alpha^4(t)\right]dt \\
&= \frac{a}{(1-\alpha^2+\alpha^4)^{3/4}} \int_0^{4K(\alpha)} \left[\frac{9\alpha^4}{2}\mathrm{sn}^4u + 3\alpha^2\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)\mathrm{sn}^2u \right. \\
&\qquad \left. + \frac{1}{2}\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)^2\right]du \\
&\quad + \frac{9(b-a)}{4(1-\alpha^2+\alpha^4)^{5/4}} \int_0^{4K(\alpha)} \left[3\alpha^6\mathrm{sn}^6u + 3\alpha^4\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)\mathrm{sn}^4u \right. \\
&\qquad \left. + \alpha^2\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)^2 \mathrm{sn}^2u + \frac{1}{9}\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)^3\right]du \\
&\quad - \frac{b}{2\left(1-\alpha^2+\alpha^4\right)^{7/4}} \int_0^{4K(\alpha)} \left[\frac{81\alpha^8}{4}\mathrm{sn}^8u + 27\alpha^6\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)\mathrm{sn}^6u \right. \\
&\qquad + \frac{27\alpha^4}{2}\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)^2 \mathrm{sn}^4u + 3\alpha^2\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)^3 \mathrm{sn}^2u \\
&\qquad \left. + \frac{1}{4}\left(\sqrt{1-\alpha^2+\alpha^4} - 1 - \alpha^2\right)^4\right]du
\end{aligned}
$$

$$= \frac{6a}{5\left(1-\alpha^2+\alpha^4\right)^{5/4}}\left[2\left(\alpha^4-\alpha^2+1\right)E(\alpha)-\left(\alpha^4-3\alpha^2+2\right)K(\alpha)\right]$$

$$- \frac{6b}{35\left(1-\alpha^2+\alpha^4\right)^{7/4}}\left\{\left[7\left(\alpha^4-3\alpha^2+2\right)\sqrt{\alpha^4-\alpha^2+1}+5\left(\alpha^6+\alpha^4-4\alpha^2+2\right)\right]K(\alpha)\right.$$

$$\left. -\left[5\left(2\alpha^6-3\alpha^4-3\alpha^2+2\right)+14\left(\alpha^4-\alpha^2+1\right)^{3/2}\right]E(\alpha)\right\}$$

where we have used the formulas for the integrals of even powers of sn(u) given in Section 4.10 and the formula

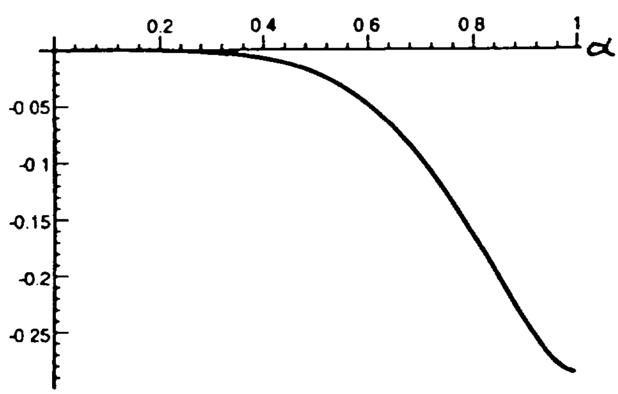
$$\int_0^{4K(\alpha)} \mathrm{sn}^8(u)dn = \frac{6\left(1+\alpha^2\right)}{7\alpha^2}\int_0^{4K(\alpha)} \mathrm{sn}^6 u du - \frac{5}{7\alpha^2}\int_0^{4K(\alpha)} \mathrm{sn}^4 u du$$

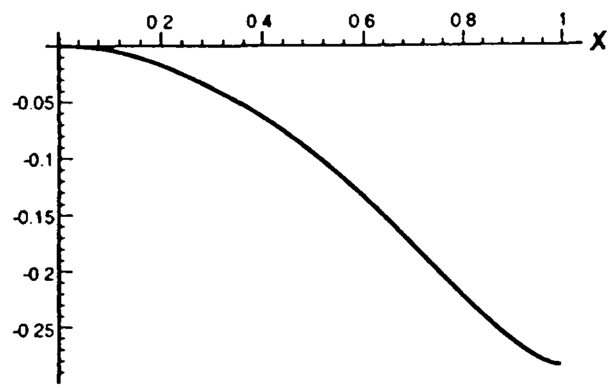
given on page 192 of [40]. It follows that $M(\alpha, \boldsymbol{\mu})$ has a simple zero iff

$$\frac{a}{b} = \frac{\left\{\left[7\left(\alpha^4-3\alpha^2+2\right)\sqrt{\alpha^4-\alpha^2+1}+5\left(\alpha^6+\alpha^4-4\alpha^2+2\right)\right]K(\alpha) -\left[5\left(2\alpha^6-3\alpha^4-3\alpha^2+2\right)+14\left(\alpha^4-\alpha^2+1\right)^{3/2}\right]E(\alpha)\right\}}{7\left(\alpha^4-\alpha^2+1\right)^{1/2}\left[2\left(\alpha^4-\alpha^2+1\right)E(\alpha)-\left(\alpha^4-3\alpha^2+2\right)K(\alpha)\right]}.$$

This function is plotted, using Mathematica, as a function of α and as a function of x below, where from the expression for $x_\alpha(T_\alpha/4)$ we have

$$x = \frac{2\alpha^2-1}{2\sqrt{1-\alpha^2+\alpha^4}}+\frac{1}{2} \quad \text{and} \quad 2\alpha^2 = 1+(2x-1)\sqrt{\frac{3}{3+4x-4x^2}}.$$





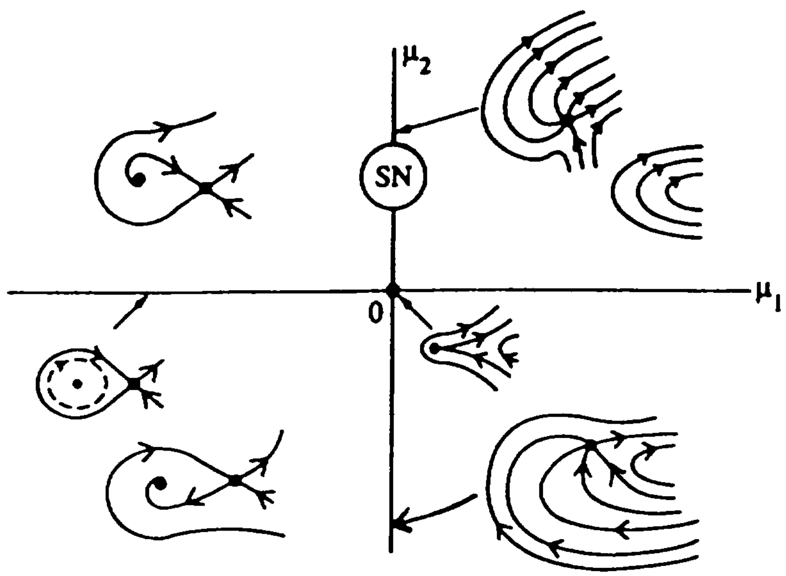
From the graphs, we see that for $b > 0$ and sufficiently small $\varepsilon > 0$ the above system has exactly one limit cycle for $-.28\cdots < a/b < 0$. As in the first part of this section, it is easy to show that the system in this problem has a subcritical Hopf bifurcation in which an unstable limit cycle bifurcates from the origin as a decreases from zero; according to the above graphs, it expands monotonically as a decreases from zero to $(-.28\cdots)b$. Computing the Melnikov function along the homoclinic loop shows that this system has a homoclinic loop for $a(\varepsilon) = -2b/7 + O(\varepsilon) = (-.28\cdots)b + O(\varepsilon)$. And for $v_1 = -1/4$, the homoclinic loop bifurcation value $\alpha = a = v_2 - 1/2 = -2/7b + O(\varepsilon)$ for $b = \beta/2 = 1/2$ corresponds to $v_2 = 5/14 + O(\varepsilon)$ as computed earlier in this section.

4. For the system (6), $Df(\mathbf{x}) = [0, 1; \mu_1 + 3x^2 - 2xy, \mu_2 - x^2]$. For $\mu_1 \geq 0$, the origin is the only critical point and $Df(\mathbf{0}) = [0, 1; \mu_1, \mu_2]$; thus, for $\mu_1 > 0$, the origin is a saddle and for $\mu_1 = 0$ the flow on the center manifold, $y = -x^3/(3\mu_2) + O(x^4)$, is given by $\dot{x} = -x^3/(3\mu_2) + O(x^4)$ and we have a topological saddle at the origin. For $\mu_1 < 0$ we have critical points at $(0, 0)$ and $\left(\pm\sqrt{-\mu_1},\ 0\right)$; the origin is a source for $\mu_2 > 0$ and a sink for $\mu_2 \leq 0$ where we must use equation (3′) in Section 4.4 in order to determine the stability of the origin when $\mu_2 = 0$; since $Df\left(\pm\sqrt{-\mu_1},\ 0\right) = [0, 1; -2\mu_1, \mu_1 + \mu_2]$, these critical points are both saddles. Using equation (4) in Section 4.2, we can show that there is a pitchfork bifurcation for $\mu_1 = 0$ (and $\mu_2 \neq 0$) in which three critical points bifurcate from the origin as μ_1 decreases through $\mu_1 = 0$. Since for $\mu_1 < 0$ and $\mu_2 = 0$ the origin is stable, it follows from Theorem 1 in Section 4.4 that there is a supercritical Hopf bifurcation in which a stable limit cycle bifurcates from the origin as μ_2 increases from zero. It then follows from the theory of rotated vector fields in Section 4.6 that for $\omega = -1$ this stable limit cycle expands as the parameter μ_2 increases until it intersects both of the saddles at $\left(\pm\sqrt{-\mu_1},\ 0\right)$ at some homoclinic (or heteroclinic) loop bifurcation value $\mu_2 = h(\mu_1) = -\mu_1/5 + O\left(\mu_1^2\right)$; this approximation was derived in this section using the results of Example 3 in Section 4.10; we have also used the symmetry of this system about the origin to deduce that the expanding limit cycle intersects both saddle points simultaneously. The bifurcation set and phase portraits for this problem are shown in Figures 7.3.4 and 7.3.5 in [G/H].

5. The details in Problem 5 are similar to the details in Problem 4 except that in Problem 5 we use the results of Theorem 5 and Problem 6 in Section 4.10 to establish the results concerning the limit cycles for this problem. We also make use of the symmetry of this system about the origin. (See p. 146 in the appendix.)

6. If we make the transformation of coordinates $(x, y, t, \mu_1, \mu_2) \to (x, -y, -t, \mu_1, -\mu_2)$, the system in this problem is transformed into the system (2). Hence, the bifurcation set and corresponding phase portraits are obtained from Figure 3 by rotating the μ_1, μ_2 plane through 180° about the μ_1-axis and by rotating the phase portraits through 180° about the x-axis and reversing the arrows.

7. For $\mu_1 > 0$ there are no critical points and for $\mu_1 \le 0$ there are critical points at $\left(\pm\sqrt{-\mu_1},\ 0\right)$. $Df\left(\pm\sqrt{-\mu_1},\ 0\right) = \left[0,\ 1;\ \pm 2\sqrt{-\mu_1}, \mu_2\right]$; for $\mu_1 < 0$, $\left(2\sqrt{-\mu_1},\ 0\right)$ is a saddle and $\left(-2\sqrt{-\mu_1},\ 0\right)$ is a sink if $\mu_2 < 0$ and a source if $\mu_2 > 0$; for $\mu_2 = 0$, by the symmetry with respect to the x-axis, $\left(-2\sqrt{-\mu_1},\ 0\right)$ is a center. For $\mu_1 = \mu_2 = 0$, there is one non-hyperbolic critical point at the origin; and from Theorem 3 in Section 2.11 it is a cusp. For $\mu_1 = 0$ and $\mu_2 \ne 0$, according to Theorem 1 in Section 2.11, there is a saddle-node at the origin. For $\mu_1 = 0$ and $\mu_2 \ne 0$, there is a single zero eigenvalue and according to Theorem 1 in Section 4.2 there is a saddle-node bifurcation, viz.:



PROBLEM SET 4.14

1. Setting $\tau(x^+, y^+) = 0$ leads to $c(\alpha + S) = 1 + \alpha\beta + \alpha y^+$ with $y^+ = (\alpha - \beta + S)/2$ and $S = \sqrt{(\alpha-\beta)^2 - 4\gamma^2}$; and this in turn yields $c = [1 + \alpha(\alpha + \beta + S)/2]/(\alpha + S)$ for the Hopf bifurcation surface H^+. For $\gamma = 0$ and $\alpha > \beta$ this reduces to the Hopf bifurcation surface $H^+ : c = (1 + \alpha^2)/(2\alpha - \beta)$ for (4). In order to show that for $\tau(x^+, y^+) = 0$, the critical point (x^+, y^+) is a stable weak focus (and to complete part b of this problem), rather than using equation (3′), it is easier to use the following formula for the Liapunov number, σ, of the quadratic system $\dot{x} = -x + Ey + y^2$, $\dot{y} = Fx + y - xy + cy^2$ with $1 + EF < 0$ given in Lemma 2.3 in [51]: $\sigma = F\left[cF^2 + (cF+1)(F - E + 2c)\right]k$ with the positive constant $k = 3\pi/[2|EF|\,|1 + EF|^{3/2}]$. (This formula also follows directly from eq. (3′) in Section 4.4 for the above quadratic system. Cf. Problem 9 in Section 4.4.) For $E = \beta + 2y^+$ and $F = \alpha - c - y^+$ we see that $\sigma = 0$ iff $2(2S - \beta)c^2 + [(\alpha + \beta - S)(\beta - 2S) + 2]c + (\beta - \alpha - 3S) = 0$, where we have used equation (8) for y^+. Solving this quadratic for c then leads to the equation of the H_2^+ surface given in part (b). (Numerical computation shows that the solution with the minus sign has no intersection with the H^+ surface and it can therefore be ignored.) And then for $\gamma = 0$ and $S = \alpha - \beta$, it can be shown that the discriminant $b^2 - 4ad < 0$, i.e., that σ does not change sign and that for $\alpha = 2\beta = c = 1$, σ is negative. Thus, for $\gamma = 0$, a supercritical Hopf bifurcation occurs as c increases.

2. The system (5) experiences a Takens-Bogdanov bifurcation at the origin when both $\delta(0, 0) = \gamma^2 = 0$ and $\tau(0, 0) = -1 + c\beta - \alpha\beta - \gamma^2 = 0$, i.e., when $\gamma = 0$ and $c = \alpha + 1/\beta$; the linear part of (5) at the origin then has the form $Df(0, 0) = [-1, \beta; -1/\beta, 1]$ with $\beta \neq 0$.

3. For $\alpha = \beta + 2\gamma$, $S = 0$ and $\delta(x^\pm, y^\pm) = \pm S y^\pm = 0$ and since $y^\pm = (\alpha - \beta)/2$, $\tau(x^\pm, y^\pm) = \alpha c - 1 - \alpha(\alpha + \beta)/2 = (\beta + 2\gamma)c - 1 - (\beta + 2\gamma)(\beta + \gamma)$ for $\alpha = \beta + 2\gamma$; it follows that $\tau(x^\pm, y^\pm) \neq 0$ if $c \neq \beta + \gamma + 1/(\beta + 2\gamma)$. We have for $\alpha = \beta + 2\gamma$ that the matrix $A \equiv Df(x^\pm, y^\pm) = [-1, \alpha; \beta + \gamma - c, \alpha(c - \beta - \gamma)]$ has one zero eigenvalue with eigenvector $\mathbf{v} = (\alpha, 1)^T$ and one non-zero eigenvalue with eigenvector $\mathbf{v}_2 = (1, c - \beta - \gamma)^T$. Transforming the linear part to

Jordan canonical form (and normalizing the time) we get $B = P^{-1}AP = [1, 0; 0, 0]$ and then it can be shown that the transformed system $\dot{x} = (0, \mu)^T + Bx + Q_2(x)$ satisfies the conditions of Theorem 1 in Section 4.2.

4. For $\alpha = \beta + 2\gamma$, $A = Df(x^\pm, y^\pm) = [-1, \alpha; \beta + \gamma - c, \alpha(c - \beta - \gamma)]$ as in Problem 3. Then for $c = \beta + \gamma + 1/(\beta + 2\gamma) = \beta + \gamma + 1/\alpha$, it follows that $A = Df(x^\pm, y^\pm) = [-1, \alpha; -1/\alpha, 1]$ and that $\delta(x^\pm, y^\pm) = \tau(x^\pm, y^\pm) = 0$; i.e., we have a double-zero eigenvalue bifurcation occurring in this case, and from the results in Section 4.12, it follows that the quadratic system (5) experiences a Takens-Bogdanov bifurcation.

5. Setting $\tau(0, 0) = 0$ leads to $c = \alpha + (1 + \gamma^2)/\beta$, the Hopf bifurcation surface, H^0, for the critical point of (5) at the origin. And then, as in Problem 1, using the simplified formula for the Liapunov number $\sigma = F[cF^2 + (cF + 1)(F - E + 2c)]k$, from [51], with $E = \beta$ and $F = \alpha - c$, we find that $\sigma = 0$ iff $(\beta - 2\alpha)c^2 + (2\alpha^2 - \alpha\beta + 1)c + \alpha - \beta = 0$. And this leads to the formula for the multiplicity-two Hopf bifurcation surface given in Problem 5. (Numerical computation shows that the solution of the above quadratic with the minus sign has no intersection with the H^0 surface and it is therefore disregarded.)

6. The global existence and analyticity of the homoclinic loop bifurcation surface HL^+ for the system (5) follow from the theory of rotated vector fields in Section 4.6 and the uniqueness of analytic continuations. The system (5) defines a semi-complete family of rotated vector fields mod $x = \beta y + y^2$ with parameter $c \in (-\infty, \infty)$ according to the definition in Section 4.6. Let us consider the case when the Liapunov number σ in Problem 1 is negative. (The case when $\sigma > 0$ is treated in a similar manner.) First of all, it follows from Theorem 5 in Section 4.6 that for $\sigma < 0$ a unique limit cycle is born in a Hopf bifurcation at a value of $c \in H^+$; and then according to Theorems 1 and 4 in Section 4.6, that limit cycle expands monotonically as c increases until it intersects the saddle point P^- and forms a homoclinic loop at some value of $c = h(\alpha, \beta, \gamma)$. (It follows

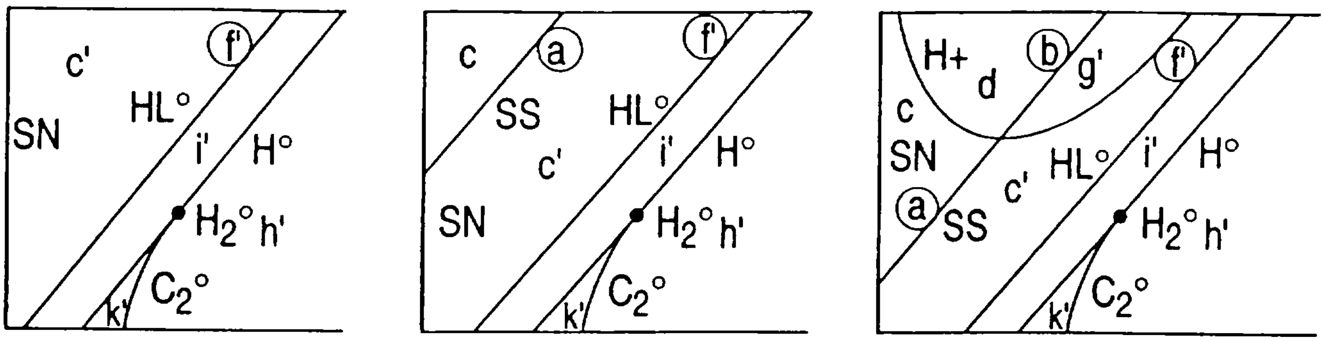
from the Poincaré-Bendixson theorem that the outer boundary of this one-parameter family of limit cycles cannot include the saddle point at (1, 0, 0) on the equator of the Poincaré sphere.) Thus, for each point $(\alpha, \beta, \gamma) \in R$, defined in Theorem 5, there exists a unique value of c, $c = h(\alpha, \beta, \gamma)$ at which (5) has a homoclinic loop at the saddle P^-. The analyticity of the function $h(\alpha, \beta, \gamma)$ then follows from the stable manifold theorem and the implicit function theorem for analytic functions as in [38]. A similar type of analysis can be used to establish the existence of the analytic surfaces HL^0, C_2^+, and C_2^0; the serious student should see Remark 10 in [38] regarding the existence of these latter two surfaces.

7. According to Theorem 1 and Remark 1 in Section 4.8, (5) has a multiple homoclinic loop bifurcation for the homoclinic loop at the saddle point P^- if $\sigma = \tau(x^-, y^-) = -1 + \beta(c - \alpha) + (2c - \alpha)y^- = 0$; and then since $y^- = [\alpha - \beta - S]/2$, it follows that $\sigma = 0$ iff $c = [1 + \alpha(\alpha + \beta - S)/2]/(\alpha - S)$.

8. The details of this problem are similar to those in Problem 3 and are left to the student to carry out.

9. Following the hint given in this problem, we see that the H^0 and HL^0 bifurcation curves enter the region E for $\beta < 0$ and for points on the HL^0 curve we have the phase portrait (f′) given in Figure 8; furthermore, for points between the HL^0 and H^0 curves, we have the phase portrait (i′) in Figure 8; and for points on the H^0 curve and to the right of the H^0 curve, we have the phase portrait (h′) shown in Figure 8. Finally, there is one last feature, not shown in Figure 18, that occurs for $\beta < 0$: as in the figures shown below, there is a C_2^0 bifurcation curve of multiplicity-two limit cycles emanating from the H_2^0 point on the H^0 curve (determined in Problem 6 above) and for points between the H^0 curve and the C_0^2 curve we have the phase portrait (k′) in Figure 8. Thus, the charts 1, 2 and 3 in Figure 16

are transposed into the charts 1, 2 and 3 shown below for $-1 << \beta < 0$ (and a similar transposition occurs for charts 4 and 5 in Figure 16).



PROBLEM SET 4.15

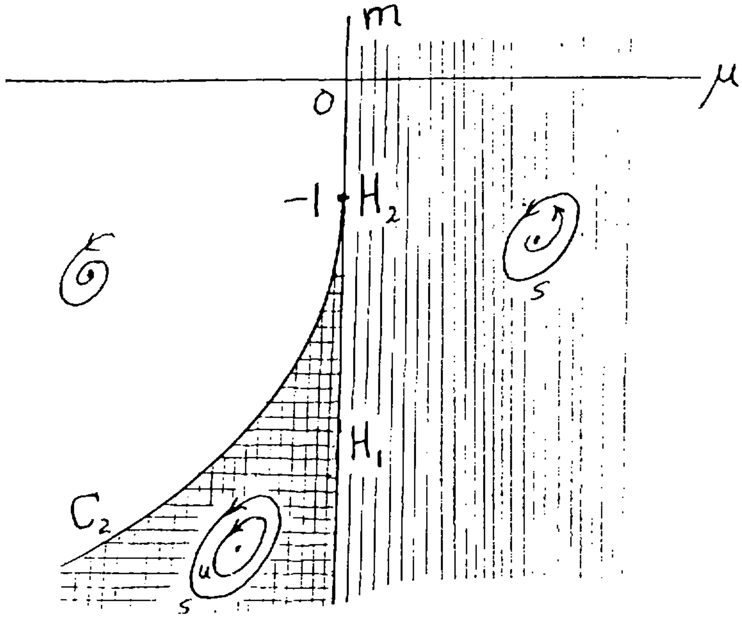
1. From equation (8) in Section 4.14, as $\gamma \to 0$, $y^+ \to \alpha - \beta$, $y^- \to 0$, $x^+ \to \alpha(\alpha - \beta)$ and $x^- \to 0$; $\delta(0, 0) = \gamma^2 \to 0$; and $\tau(0, 0) = -1 + \beta(c - \alpha) - \gamma^2 \to -1 + \beta(c - \alpha)$. Regarding Sotomayor's Theorem for (2): Since $\delta(0, 0) = 0$ and $\tau(0, 0) = -1 + \beta(c - \alpha) \neq 0$, the matrix $A = Df(\mathbf{0}, 0) = [-1, \beta; (\alpha - c), \beta(c - \alpha)]$ has a simple eigenvalue $\lambda = 0$ with corresponding eigenvector $\mathbf{v} = (\beta, 1)^T$; the matrix A^T has an eigenvector $\mathbf{w} = (\alpha - c, 1)^T$ corresponding to $\lambda = 0$. Thus, the conditions $\mathbf{w}^T\, \mathbf{f}_\mu(\mathbf{0}, 0) = (\alpha - c, 1) \cdot (0, 1) = 1 \neq 0$ and $\mathbf{w}^T[D^2\mathbf{f}(\mathbf{0}, 0)(\mathbf{v}, \mathbf{v})] = (\alpha - c, 1) \cdot (2, -2\beta + 2c) = 2(\alpha - \beta) \neq 0$ are both satisfied for the system (2) in Theorem 1. They are similarly shown to hold for the system (5) in Theorem 1′. Thus, according to Sotomayor's Theorem, the system (2) in Theorem 1 experiences a saddle-node bifurcation (of codimension 1) at the critical point at the origin at the bifurcation value $\mu = 0$ and the system (5) in Theorem 1′ experiences a saddle-node bifurcation (of codimension 1) at the critical point P^+ at the bifurcation value $\mu = 0$.

2. Applying the linear transformation $x = u + \beta v$, $y = (c - \alpha)u + v$ or equivalently $u = (x - \beta y)/\delta$, $v = [(\alpha - c)x + y]/\delta$ with $\delta = 1 - \beta(c - \alpha)$, together with $t \to -\delta t$, to the system (1) with $\gamma = 0$, we find $\dot u = (\dot x - \beta\dot y)/\delta = u + a_{20}u^2 + a_{11}uv + a_{02}v^2$ and $\dot v = [(\alpha - c)\dot x + \dot y]/\delta$

$= b_{20}u^2 + b_{11}uv + b_{02}v^2$ where $a_{20} = (\alpha - c)(\alpha\beta c - \alpha + \beta - \beta c^2 + c)/\delta^2$, $a_{11} = (-2c + 2\beta c^2 - \beta + 2\alpha - 2\alpha\beta c + \alpha\beta^2 - \beta^2 c)/\delta^2$, $a_{02} = (1 + \beta^2 - \beta c)/\delta^2$, $b_{20} = (\alpha - c)(\alpha^2 - \alpha c + 1)/\delta^2$, $b_{11} = (2\alpha c - \beta c - 1 - 2\alpha^2 + \alpha\beta)/\delta^2$ and $b_{02} = (\beta - a)/\delta^2$. We see that for $\alpha = \beta$, $b_{02} = 0$. Thus, on the center manifold, $u = -a_{02}v^2 + 0(v^3)$, we have $\dot{v} = -b_{11}a_{02}v^3 + 0(v^4)$ or $\dot{v} = -v^3 + 0(v^4)$ after rescaling the time. Similarly, applying the above-mentioned linear transformation to the system (6), we find that the flow on the center manifold is determined by $\dot{v} = \mu_1 + \mu_2 v - v^3 + 0(\mu_1 v, \mu_1^2, \mu_2^2, v^4, \cdots)$; cf. the proof of Theorem 3.4 and Remark 3.5 in [60].

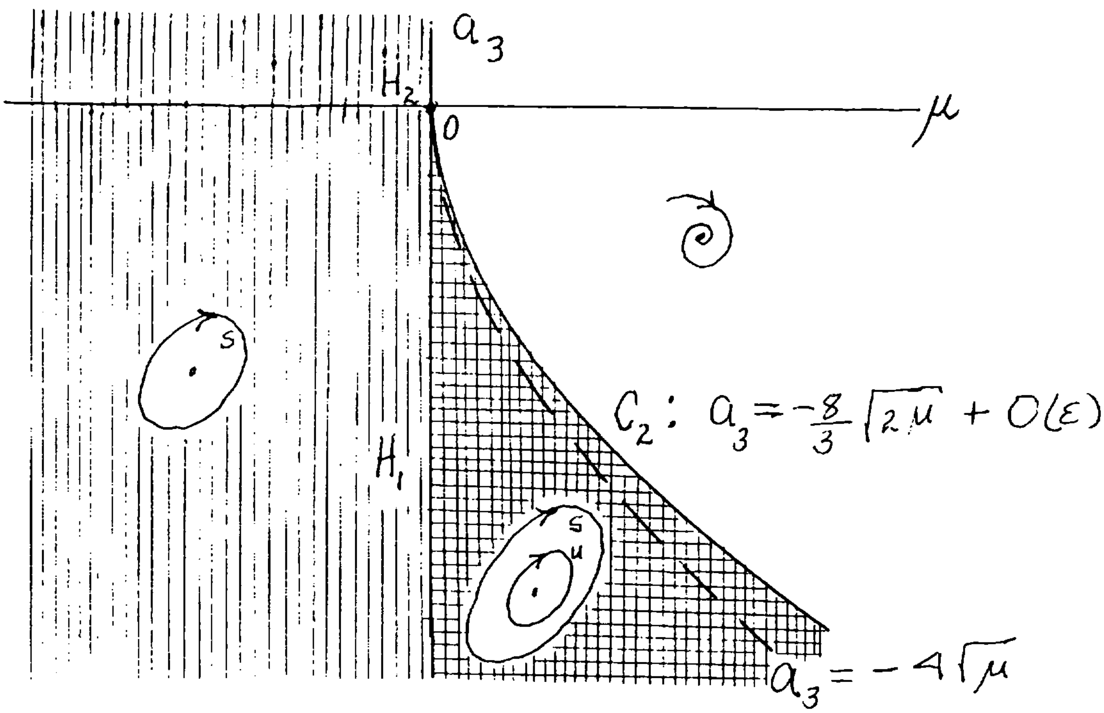
3. Applying the linear transformation of coordinates $x = (u - v)/(\alpha c - \alpha^2 - 1)$, $y = (c - \alpha)u/(\alpha c - \alpha^2 - 1)$ or equivalently $u = (\alpha c - \alpha^2 - 1)y/(c - \alpha)$, $v = (\alpha c - \alpha^2 - 1)[y/(c - \alpha) - x]$ to the system (1) with $\gamma = 0$, $\alpha \neq \beta$, $\beta(c - \alpha) = 1$ and $\beta \neq 2c$, we find $\dot{u} = v + au^2 + buv$ and $\dot{v} = u^2 + euv$ with $a = (c^2 - \alpha c - 1)/(\alpha c - \alpha^2 - 1)$ and $b = e = 1/(\alpha c - \alpha^2 - 1)$ as in the proof of Lemma 3.7 in [60]. We therefore have that $e + 2a = -(1 - 2c^2 + 2\alpha c)/(\alpha c - \alpha^2 - 1) \neq 0$ iff $1 - 2c^2 + 2\alpha c = 1 - 2c(c - \alpha) = 1 - 2c/\beta \neq 0$, i.e., iff $\beta \neq 2c$.

4. (a) From the results of Problem 8 (a, b) in Problem Set 4.4, we have that, for $\mu = 0$, $W_1 = -1 - m$ and $W_2 = 2(m + 2)(m - 3)$ for $n = 0$ (and $a = b = \ell = 1$). Thus, for $\mu = 0$ and $m = -1$, $W_1 = 0$ and $W_2 = -8 < 0$ so that, according to Theorem 4 in Section 4.4, the system in this problem has a weak focus of multiplicity 2 at the origin and it is stable. Since $\det(P, Q; P_\mu, Q_\mu) = -r^2 + 0(r^3) < 0$ in a neighborhood of the origin, this sytem defines a family of negatively rotated vector fields with parameter μ in a neighborhood of the origin, according to Definition 1 in Section 4.6. Since $\dot{\theta} > 0$ near the origin (i.e., $\omega = +1$), and $\sigma = 3\pi W_1/2 < 0$ for $m > -1$ and $\sigma > 0$ for $m < -1$ (and $W_1 = 0$ while $W_2 < 0$ if $m = -1$), it follows from Theorem 5 and Figure 1 in Section 4.6 (or Theorem 1 in Section 4.4 for $m \neq -1$) that a stable, positively oriented limit cycle bifurcates from the origin as μ increases if $m \geq -1$ and that an unstable, positively oriented limit cycle bifurcates from the origin as μ decreases if $m < -1$. According to Theorem 2 in Section 4.1, for $m = -1$ and a fixed $\mu > 0$, there exists a $\delta > 0$ such that the hyperbolic (stable) limit cycle, which bifurcates from the origin as μ increases from zero, continues to exist for $m < -1$ and $|m + 1| < \delta$. For such a fixed $m = m_0 < -1$, according to Theorems 1, 2, and 6 in Section 4.6, the abovementioned stable, positively oriented limit cycle contracts as μ decreases until it intersects the unstable limit cycle generated in a Hopf bifurcation at $\mu = 0$ (which expands as μ decreases from zero) at some value of $\mu = \mu_0 < 0$ and forms a multiplicity-2 limit cycle. This defines a point (μ_0, m_0) on the multiplicity-2 limit cycle bifurcation curve C_2 which, according to the results in [38], is an analytic curve which intersects the Hopf bifurcation curve $\mu = 0$ (i.e., the m-axis) tangentially at the point $(\mu, m) = (0, -1)$. This leads to the following bifurcation set similar to Figure 2 in Section 4.15:



(b) For $\mu = a_3 = 0$, it follows from equation (3′) in Section 4.4 that $\sigma = 0$; hence, this system has a weak focus of multiplicity $m \geq 2$ at the origin. But according to Theorem 5 in Section 3.8, $m \leq 2$; hence, $m = 2$. For $\mu = a_3 = 0$, $\dot{r} = -16\varepsilon x^6/5r < 0$ for $\varepsilon > 0$ and $x \neq 0$; hence, the origin is a stable focus. For $\mu = 0$ and $a_3 \neq 0$, equation (3′) in Section 4.4 implies that $\sigma = -3\pi a_3\varepsilon/2$. Also, $\dot{\theta} < 0$ near the origin (i.e., $\omega = -1$) and according to Definition 1 in Section 4.6, for $\varepsilon > 0$ this system defines a family of negatively rotated vector fields (mod $x = 0$) with parameter μ since $[P, Q; P_\mu, Q_\mu] = -\varepsilon x^2$. Thus, if we let σ denote the stability of the origin, it follows that $\sigma < 0$ for $a_3 \geq 0$ and $\sigma > 0$ for $a_3 < 0$. It therefore follows from Theorem 5 and Figure 1 in Section 4.6 (or from Theorem 1 in Section 4.4 for $a_3 \neq 0$) that a stable, negatively oriented limit cycle bifurcates from the origin as μ decreases from zero if $a_3 \geq 0$ and that an unstable, negatively oriented limit cycle bifurcates from the origin as μ increases from zero if $a_3 < 0$. According to Theorem 2 in Section 4.1, for $a_3 = 0$ and for a fixed $\mu < 0$, there exists a $\delta > 0$ such that the hyperbolic (stable) limit cycle, which bifurcates from the origin as μ decreases from zero, continues to exist for $a_3 < 0$ and $|a_3| < \delta$. For such a fixed $a_3 = a_3^0 < 0$, according to

Theorems 1, 2, and 6 in Section 4.6, the abovementioned stable, negatively oriented limit cycle contracts as μ increases until it intersects the unstable limit cycle generated in a Hopf bifurcation (which expands as μ increases from zero) at some value of $\mu = \mu_0 > 0$ and forms a multiplicity-2 limit cycle. This defines a point (μ_0, a_3^0) on the multiplicity-2 limit cycle bifurcation curve C_2 which, according to the results in [38], is an analytic curve which intersects the Hopf bifurcation curve $\mu = 0$ (i.e., the a_3 axis) tangentially at the origin of the (μ, a_3) plane. This leads to the following bifurcation set similar to Figure 2 in Section 4.15:



Finally, for the perturbed system (with small $\varepsilon \neq 0$), we can compute the Melnikov function as in Example 2 in Section 4.9 to approximate the shape of the C_2 bifurcation curve for small $\varepsilon > 0$: For $\gamma_\alpha(t) = (\alpha \cos t, \alpha \sin t)^T$, $T_\alpha = 2\pi$ and $\boldsymbol{\mu} = (\mu, a_3)$,

$$
\begin{aligned}
M(\mu, \alpha) &= \int_0^{T_\alpha} f(\gamma_\alpha(t)) \wedge g(\gamma_\alpha(t), \mu)dt \\
&= -\int_0^{2\pi}\left[\mu x_\alpha^2(t) + a_3 x_\alpha^4(t) + \frac{16}{5} x_\alpha^6(t)\right]dt \\
&= -\int_0^{2\pi}\left[\mu\alpha^2\cos^2 t + a_3\alpha^4\cos^4 t + \frac{16}{5}\alpha^6\cos^6 t\right]dt \\
&= -2\pi\alpha^2\left[\frac{\mu}{2} + \frac{3}{8}a_3 + \alpha^4\right] = -\pi\alpha^2\left[\mu + \frac{3}{4}a_3\alpha^2 + 2\alpha^4\right].
\end{aligned}
$$

It follows that $M(\mu, \alpha) = -\pi\alpha^2\left(\sqrt{\mu} - \sqrt{2}\alpha^2\right)^2$ iff $a_3 = -8\sqrt{2\mu}/3$, in which case for $\alpha_0 = \sqrt[4]{\mu/2}$, $M(\mu, \alpha_0) = M_a(\mu, \alpha_0) = 0$ and $M_{\alpha\alpha}(\mu, \alpha_0) = -8\pi\mu < 0$ for $\mu > 0$. Also, $M_\mu(\mu, \alpha) = -\pi\alpha^2$, i.e., $M_\mu(\mu, \alpha_0) = -\pi\sqrt{\mu/2} < 0$ for $\mu > 0$. Therefore, by Theorem 2 in Section 4.10, there exists $a_3 = -8\sqrt{2\mu}/3 + 0(\varepsilon)$ such that this system has a unique limit cycle of multiplicity 2 in an $0(\varepsilon)$ neighborhood of the circle of radius $r = \sqrt[4]{\mu/2}$ for all sufficiently small $\varepsilon > 0$; i.e., the multiplicity-2 limit cycle bifurcation curve C_2 is given by $a_3 = -8\sqrt{2\mu}/3 + 0(\varepsilon) \cong -3.77\sqrt{\mu} + 0(\varepsilon)$ for sufficiently small $\varepsilon > 0$ and we see that the curve $a_3 = -4\sqrt{\mu}$, for which Example 3 in Section 4.4 has two limit cycles, lies below the C_2 bifurcation curve in the above figure (i.e., it lies in the doubly cross-hatched region in the (μ, a_3) plane for which the system in this problem has two limit cycles which are shown in Figure 5 in Section 4.4 for $\varepsilon = .01$ and $\mu = .5$ or 1).

5. (a) According to Theorem 3 in Section 3.4, for $\mu_1 = 0$, $P(0) = 0$ and $P'(0) = 1$. Therefore, the displacement function, $d(s) = P(s) - s$, satisfies $d(0) = d'(0) = 0$ and according to the comment following Theorem 3 in Section 3.4, $d''(0) = 0$. Next, it follows from equation (3) in Section 3.4 that for $\mu_1 = 0$, $\sigma = d'''(0) = 12\pi\mu_2 = 0$ for $\mu_2 = 0$ and, once again according to the comment following Theorem 3, this implies that $d^{(IV)}(0) = 0$. Thus, according to the definition of the multiplicity, m, of a focus in Section 3.4, $m \geq 2$ for $\mu_1 = \mu_2 = 0$.

(b) Using the equations for $\dot{r}$ and $\dot{\theta}$ in Section 2.10, we find that the system in this problem can be written in polar coordinates as

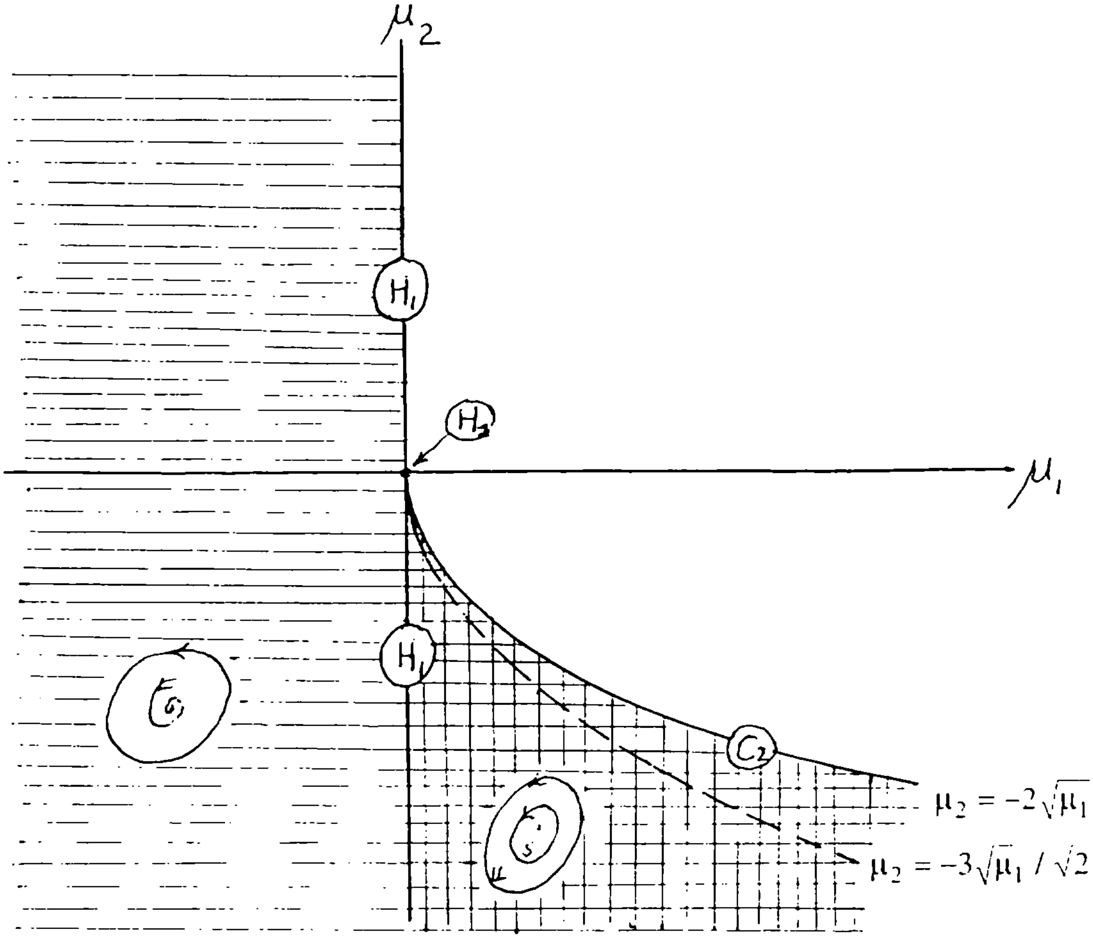
$$\dot{r} = r\left(\mu_1 + \mu_2 r^2 + r^4\right)$$

and $$\dot{\theta} = 1.$$

Thus, if $\mu_1 = \mu_2 = 0$, we obtain $dr/d\theta = r^5$. The solution of this differential equation with $r(0) = r_0$ is given by $r(\theta) = r_0\left[1 - 4r_0^4\theta\right]^{-1/4}$. The Poincaré map for the focus at the origin of this system, as defined in Section 3.4, is then given by $P(r_0) = r(2\pi) = r_0\left[1 - 8\pi r_0^4\right]^{-1/4}$. We can then compute $P'(r_0) = \left[1 - 8\pi\, r_0^4\right]^{-5/4}$, $P''(r_0) = 40\pi\, r_0^3\left[1 - 8\pi\, r_0^4\right]^{-9/4}$, $P'''(r_0) = 120\pi\, r_0^2\left[1 - 8\pi\, r_0^4\right]^{-9/4} + 0\left(r_0^6\right)$, $P^{(IV)}(r_0) = 240\pi\, r_0\left[1 - 8\pi\, r_0^4\right]^{-9/4} + 0\left(r_0^5\right)$ and $P^{(V)}(r_0) = 240\pi\left[1 - 8\pi\, r_0^4\right]^{-9/4} + 0\left(r_0^4\right)$ as $r_0 \to 0$. Thus $d^{(V)}(0) = P^{(V)}(0) = 240\pi > 0$; and this implies that the origin is an unstable, weak focus of multiplicity $m = 2$ for $\mu_1 = \mu_2 = 0$.

(c) From part (b) we see that $dr/d\theta = 0$ iff $r = 0$ or $r^4 + \mu_2 r^2 + \mu_1 = 0$. The latter equation has solutions $r^2 = \left[-\mu_2 \pm \sqrt{\mu_2^2 - 4\mu_1}\,\right]/2$ which are both positive iff $\mu_2 < 0$ and $\mu_1 > 0$. For $\mu_1 = \mu_2^2/4$ and $\mu_2 < 0$, we have one (positive) double root $r^2 = -\mu_2/2$, which corresponds to a multiplicity-2 limit cycle described by a circular orbit of radius $r = \sqrt{-\mu_2/2}$.

(d) As in part (a), it follows from equation (3) in Section 3.4 that for $\mu_1 = 0$, $\sigma = 12\pi\mu_2$. Thus, by Theorem 1 in Section 4.4, if $\mu_2 < 0$, a unique stable limit cycle bifurcates from the origin as μ_1 increases from zero and if $\mu_2 > 0$, a unique unstable limit cycle bifurcates from the origin as μ_1 decreases from zero. Next, $[P,Q; P_{\mu_1}, Q_{\mu_1}] = -r^2 < 0$ for $r \neq 0$; i.e., the system in this problem defines a one-parameter family of **negatively** rotated vector fields with parameter μ_1. Thus, by Theorem 5 in Section 4.6 and the fact that for $\mu_1 = \mu_2 = 0$ the origin is a positively oriented, unstable focus, we find that, according to the table in Figure 1 in Section 4.6 (adjusted for a negatively rotated vector field by changing the signs of $\Delta\mu$), an unstable, positively oriented limit cycle bifurcates from the origin as μ_1 decreases from zero. For $\mu_2 = 0$ and $\mu_1 < 0$, this limit cycle is described by a circular orbit of radius $r = \sqrt[4]{-\mu_1}$, according to the result in part (c); and then since $\nabla \cdot \mathbf{f} = 2\mu_1 + 6r^4 = -4\mu_1$ on this limit cycle, we have $P'(0) = e^{-8\pi\mu_1} > 1$ for $\mu_1 < 0$, according to Theorem 2 in Section 3.4; i.e., we have a **hyperbolic**, unstable limit cycle for $\mu_2 = 0$ and $\mu_1 < 0$. [See Note 1 below regarding another method for establishing the hyperbolicity of this limit cycle and those in Problem 4.] Thus, according to Theorem 2 in Section 4.1, for $\mu_2 = 0$ and for a fixed (sufficiently small) $\mu_1 < 0$, there exists a $\delta > 0$ such that the hyperbolic, unstable limit cycle, which bifurcates from the origin for $\mu_2 = 0$ as μ_1 decreases from zero, continues to exist for $-\delta < \mu_2 < 0$. Then by Theorems 1, 2 and 6 in Section 4.6, for $-\delta < \mu_2^0 < 0$, this unstable, positively oriented limit cycle contracts as μ_1 increases until it intersects the stable, positively oriented limit cycle (generated in the supercritical Hopf bifurcation, for $\mu_2 < 0$, as μ_1 increases from zero) at some value of $\mu_1 = \mu_1^0 > 0$ and forms a multiplicity-2 limit cycle. (Cf. the last figure in Figure 5 in Section 4.6.) The point $\left(\mu_1^0, \mu_2^0\right)$ lies on the multiplicity-2 limit cycle bifurcation curve C_2 which, according to the results in [38], is an analytic curve which intersects the Hopf bifurcation curve $\mu_1 = 0$ (i.e., the μ_2-axis) tangentially at the origin of the (μ_1, μ_2) plane. As was noted in part (c), $\mu_1^0 = (\mu_2^0)^2 / 4$; i.e., the multiplicity-2 limit cycle bifurcation curve, C_2, is given by $\mu_2 = -2\sqrt{\mu_1}$ for $\mu_1 > 0$. Putting all of these facts together leads to the following bifurcation set (similar to Figure 2 in this section):



(c) If we replace x by $x / \sqrt[4]{2}$ and y by $y / \sqrt[4]{2}$ in the system of differential equations

$$\dot{x} = \mu^2 x - y - 3\mu x r^2 + 2xr^4$$
$$\dot{y} = x + \mu^2 y - 3\mu y r^2 + 2yr^4$$

in Example 4 in Section 4.4, we obtain the system

$$\dot{x} = \mu^2 x - y - \frac{3}{\sqrt{2}}\mu x r^2 + xr^4$$
$$\dot{y} = x - \mu^2 y - \frac{3}{\sqrt{2}}\mu y r^2 + yr^4$$

of part (a) in this problem with $\mu_1 = \mu^2$ and $\mu_2 = -3\mu / \sqrt{2}$; i.e., with $\mu_2 = -3\sqrt{\mu_1} / \sqrt{2} \cong -2.12\sqrt{\mu_1}$ for $\mu_1 \geq 0$. This curve lies below C_2 in the region where this system has two limit cycles (as shown in the above figure). The system in Example 4 of Section 4.4 has two limit cycles described by circular orbits of radii $r = \sqrt{\mu}$ and $r = \sqrt{\mu / 2}$; i.e., the latter system above has two (unstable and stable) limit cycles described by circular orbits of radii $r = \left(\sqrt{2}\,\mu\right)^{1/2}$ and $r = \left(\mu / \sqrt{2}\right)^{1/2}$ (∗) (respectively), obtained by replacing r by $r / \sqrt[4]{2}$ in the equations $r = \sqrt{\mu}$ and $r = \sqrt{\mu / 2}$. The two equations (∗) above also follow from the equations for the radii of the two limit cycles, of the latter system above, found in part (c).

Note 1: We could also have used the theory of multiple foci and limit cycles, in Sections 4.4 and 4.5 respectively, to establish the hyperbolicity of the (unstable) limit cycle generated at the origin in Problem 5 for $\mu_2 = 0$ as μ_1 decreases from zero. The following argument, a slight variation of which also applies to Problems 4(a) and (b), should have been included in the solution to Problem 4 above:

Since for $\mu_2 = 0$ (in Problem 5), an unstable (positively oriented) limit cycle bifurcates from the origin as μ_1 decreases from zero, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $-\varepsilon_0 < \mu_1 < 0$, there exists an unstable (positively oriented) limit cycle, Γ_1, in $N_{\delta_0}(\mathbf{0})$. Since Γ_1 is unstable, it is of **odd** multiplicity $m \geq 1$ (otherwise, it would be a semi-stable limit cycle). Suppose that $m \geq 3$. Then according to Theorem 3(i) in Section 4.4, there exists an $\varepsilon > 0$ and a $\delta > 0$ (with $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$) such that any system ε-close to the system in Problem 5 with $\mu_1 = \mu_2 = 0$, in the C^{2m+1}-norm, has at most two limit cycles in $N_\delta(\mathbf{0})$ since the origin is a weak focus of multiplicity 2 according to part (b). Then for $\mu_2 = 0$ and sufficiently small $|\mu_1| < \varepsilon$, $\Gamma_1 \subset N_\delta(\mathbf{0})$ and, according to Theorem 2(ii) in Section 4.5, there exists an analytic system which is ε-close to the system in this problem with $\mu_2 = 0$ and $-\varepsilon < \mu_1 < 0$, in the C^m-norm (and we can also find an analytic system which is ε-close to the system in this problem with $\mu_2 = 0$ and $-\varepsilon < \mu_1 < 0$ in the C^{2m+1}-norm), and which has m limit cycles in $N_\delta(\mathbf{0})$. But this is a contradiction for $m \geq 3$. Thus, $m = 1$ and Γ_1 is a simple (i.e., hyperbolic) limit cycle.

Note 2: As was noted above, a variation of the above argument applies to establish the hyperbolicity of the limit cycles in Problems 4(a) and (b); however, in Problem 4(b), with $a_3 = 0$ and $\mu < 0$, we can also use the Melnikov theory to establish that the limit cycle, Γ_1, is hyperbolic: As in the solution to Problem 4(b) above, we have, for $a_3 = 0$, that $M(\alpha, \mu) = -\pi\alpha^2(\mu + 2\alpha^4)$ which implies that $M_\alpha(\alpha, \mu) = -2\pi\alpha(\mu + 6\alpha^4)$. We see that $M(\alpha, \mu) = 0$ iff $\alpha = 0$ or $\alpha = \sqrt[4]{-\mu/2}$ for $\mu < 0$; and that $M_\alpha\left(\alpha, \sqrt[4]{-\mu/2}\right) = 4\pi\mu\sqrt[4]{-\mu/2} < 0$ for $\mu < 0$. Therefore, for $a_3 = 0$ and $\mu < 0$, Γ_1 is a hyperbolic (stable) limit cycle according to Theorem 1 in Section 4.10 (since $\varepsilon > 0$ and $\omega_0 < 0$).

APPENDIX TO THE SOLUTIONS MANUAL FOR TAM 7, DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

Lawrence Perko

CONTENTS

1. LINEAR SYSTEMS

PROBLEM SET 1.2

3. (a) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\mathbf{x}$ has the solution

$$\mathbf{x}(t) = \frac{1}{3}\begin{bmatrix} 2e^{t} + e^{-2t} & e^{t} - e^{-2t} \\ 2\left(e^{t} - e^{-2t}\right) & e^{t} + 2e^{-2t} \end{bmatrix}\mathbf{x}_0$$

which implies that $x(t) = \frac{1}{3}\left(2e^{t} + e^{-2t}\right)x(0) + \frac{1}{3}\left(e^{t} - e^{-2t}\right)\dot{x}(0)$.

(b) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$. (See Problem 1(d) in Problem Set 1.1.)

$$\mathbf{x}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\mathbf{x}_0.$$

Note: This problem can also be written as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x} \text{ which has the solution}$$

$$\mathbf{x}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\mathbf{x}_0.$$

In either case, we get that the (unique) solution of the second-order differential equation in 3(b) is $x(t) = x(0)\cos t + \dot{x}(0)\sin t$.

(c) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}\mathbf{x}$ has the solution

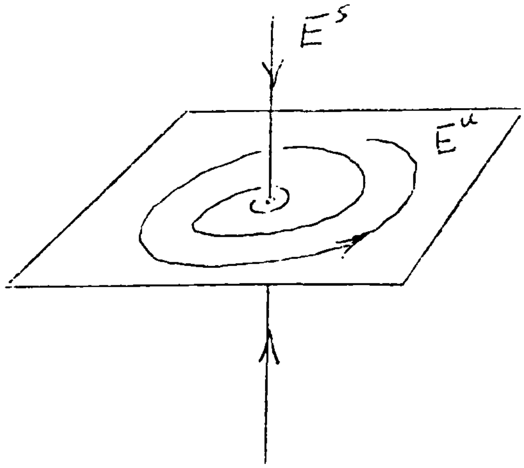
$$\mathbf{x}(t) = \frac{1}{6}\begin{bmatrix} 2(3e^{t} + e^{-t} - e^{2t}) & 3(e^{t} - e^{-t}) & -3e^{t} + e^{-t} + 2e^{2t} \\ 2(3e^{t} - e^{-t} - 2e^{2t}) & 3(e^{t} + e^{-t}) & -3e^{t} - e^{-t} + 4e^{2t} \\ 2(3e^{t} + e^{-t} - 4e^{2t}) & 3(e^{t} - e^{-t}) & -3e^{t} + e^{-t} + 8e^{2t} \end{bmatrix}\mathbf{x}_0,$$

which implies that

$$x(t) = \frac{1}{3}\left(3e^{t} + e^{-t} - e^{2t}\right)x(0) + \frac{1}{2}\left(e^{t} - e^{-t}\right)\dot{x}(0) - \frac{1}{6}\left(3e^{t} - e^{-t} - 2e^{2t}\right)\ddot{x}(0).$$

PROBLEM SET 1.6

2.



PROBLEM SET 1.8

6. (e) J = diag [1, 2, 3, 4] and the solution

$$
\mathbf{x}(t) = \begin{bmatrix} e^t & 0 & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 & 0 \\ \frac{3}{2}e^{3t} - 2e^{2t} + \frac{1}{2}e^t & 2e^{3t} - 2e^{2t} & e^{3t} & 0 \\ \frac{8}{3}e^{4t} - \frac{9}{2}e^{3t} + 2e^{2t} - \frac{1}{6}e^t & 4e^{4t} - 6e^{3t} + 2e^{2t} & 3e^{3t}(e^t - 1) & e^{4t} \end{bmatrix} \mathbf{x}_0.
$$

(g) The solution, according to the remark following Corollary 1 in Section 1.7, is given by

$$
\mathbf{x}(t) = e^{2t} \begin{bmatrix} 1 & t & 4t + t^2/2 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_0.
$$

Note: The solution to Problem 6(h) could also be obtained in this manner.

2. NONLINEAR SYSTEMS: LOCAL THEORY

PROBLEM SET 2.1

1. $$Df(\mathbf{x}) = \begin{bmatrix} 1+x_2^2+x_3^2 & 2x_1x_2 & 2x_1x_3 \\ -1+x_2x_3 & 1-x_3+x_1x_3 & -x_2+x_1x_2 \\ -2x_1 & 1 & 1 \end{bmatrix},$$

$$Df(\mathbf{0}) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad Df(0,-1,1) = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

PROBLEM SET 2.2

3. As in the proof of the Fundamental Existence Theorem (FET) in this section, we have that there exist positive constants ε and K such that $|\mathbf{f}(\mathbf{x},t)-\mathbf{f}(\mathbf{y},t)| \le K|\mathbf{x}-\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y}, \in N_\varepsilon(\mathbf{x}_0) \subset E$ and for t in some interval $(-t_0, t_0)$. Let $N_0 = \{(\mathbf{x}, t) \,|\, |\mathbf{x}-\mathbf{x}_0| \le \varepsilon/2, |t| \le t_0/2\}$ and let $M = \max|\mathbf{f}(\mathbf{x},t)|$ on the compact set N_0. Let the successive approximations $\mathbf{u}_k(t)$ be defined as stated in this problem and let $b = \varepsilon/2$. Then $b > 0$ and assuming that there exists an $a > 0$ such that $\mathbf{u}_k(t)$ is defined and continuous on $[-a, a]$ and satisfies

$$\max_{[-a,a]} |\mathbf{u}_k(t)-\mathbf{x}_0| \le b, \tag{$*$}$$

it follows exactly as in the proof of the FET that $\mathbf{u}_{k+1}(t)$ is defined and continuous on $[-a, a]$ and satisfies $|\mathbf{u}_{k+1}(t)-\mathbf{x}_0| \le Ma$ for $t \in [-a, a]$. Thus, choosing $0 < a \le \min\{b/M, t_0/2\}$, it follows by induction that $\mathbf{u}_k(t)$ is defined and continuous and satisfies ($*$) for all $t \in [-a, a]$ and $k = 1, 2, 3 \cdots$. Then since $\mathbf{u}_k(t) \in N_0$ for $t \in [-a, a]$ and $k = 1, 2, 3 \cdots$, by exactly the same sequence of estimates as in the FET (with $\mathbf{f}(\mathbf{x}, t)$ in place of $\mathbf{f}(\mathbf{x})$), we obtain that for any $\varepsilon > 0$ there exists an integer N such that for $m, n \ge N$, $\|\mathbf{u}_m - \mathbf{u}_k\| < \varepsilon$; i.e. $\{\mathbf{u}_k\}$ is a Cauchy sequence in $C([-a, a])$. Thus, $\mathbf{u}_k(t)$ converges uniformly to a continuous function $\mathbf{u}(t)$ on $[-a, a]$. The remainder of the proof follows exactly as in the proof of the FET.

4. Let A(t) be an nxn continuous matrix valued function on $[-a_0, a_0]$ and let the successive approximations to the fundamental matrix solution be defined as stated in this problem. Then using the matrix norm (defined in Section 1.3), we have $\|\Phi_1(t) - I\| \le \int_0^{|t|} \|A(t)\|\, ds \le M_0 a_0$ for $t \in [-a_0, a_0]$. It then follows easily by induction that $\|\Phi_{j+1}(t) - \Phi_j(t)\| \le (M_0 a_0)^{j+1}$ for $j = 0, 1, 2, \cdots$. And then for any positive integer N and $m > k \ge N$ we have $\|\Phi_m(t) - \Phi_k(t)\| \le \sum_{j=N}^{\infty} (M_0 a_0)^{j+1} \le (M_0 a)^{N+1} / (1 - aM_0)$ if the positive number $a \le a_0$ and $a < 1/M_0$. Thus, $\{\Phi_k\}$ is a Cauchy sequence of continuous nxn matrices on $[-a, a]$, a complete metric space. Therefore $\Phi_k(t)$ converges uniformly to a continuous nxn matrix function $\Phi(t)$ on $[-a, a]$. The remainder of the proof follows as in the proof of the Fundamental Existence Theorem in this section.

PROBLEM SET 2.3

2. (b) $u_1(t, y) = y_1 / (1 - y_1 t)$, $u_2(t, y) = \left(\frac{1}{y_1} + \frac{1}{y_2} - 1\right) e^t + t + 1 - \frac{1}{y_1}$.

$$\Phi(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = \begin{bmatrix} (1 - y_1 t)^{-2} & 0 \\ (1 - e^t)/y_1^2 & e^t \end{bmatrix},\quad Df(\mathbf{x}) = \begin{bmatrix} 2x_1 & 0 \\ -x_1^{-2} & 1 \end{bmatrix},\quad \dot{\Phi}(t) = \begin{bmatrix} 2y_1/(1 - y_1 t)^3 & 0 \\ -e^t y_1^{-2} & e^t \end{bmatrix} =$$

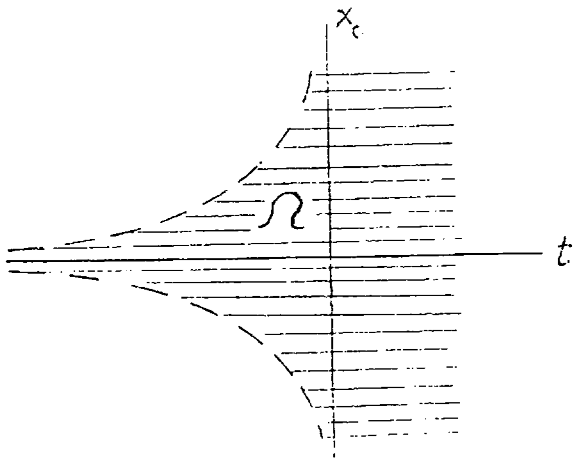
$Df[\mathbf{u}(t, \mathbf{y})]\,\Phi(t)$.

3. In this problem, the proof follows exactly as in the proof of Problem 3 in Section 2.2, contained in this supplement, with $\mathbf{f}(t, \mathbf{x}, \boldsymbol{\mu})$ in place of $\mathbf{f}(t, \mathbf{x})$.

4. The proof follows exactly as in the proof of Problem 4 in Section 2.2, contained in this supplement.

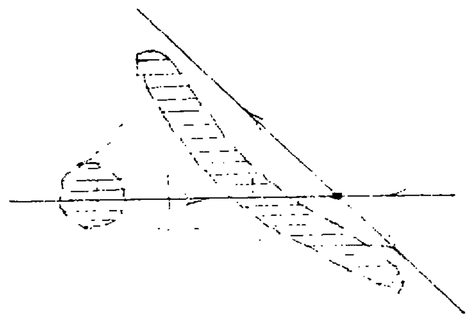
PROBLEM SET 2.5

2. $x(t) = x_0\left(1+2x_0^2t\right)^{-1/2}$



4. $\varphi_t(\mathbf{x}) = \begin{bmatrix} e^{-t} & e^{-t}-e^{2t} \\ 0 & e^{2t} \end{bmatrix}\mathbf{x}$

(See Figure 2 in Section 1.2.)



PROBLEM SET 2.8

4. $y_1(t) = y_{10}e^{-t}$, $y_2(t) = y_{20}e^{-t} + y_{10}^2 z_0 te^{-t}$, $z(t) = z_0e^t$, $\Psi_k(\mathbf{y}, z) = z$ for $k = 0, 1, 2 \cdots$ and $\Psi(\mathbf{y}, z) = z$. By either (3) or (6) we obtain $\Phi_k(\mathbf{y}, z) = \left(y_1, y_2 \pm ky_1^2 z\right) \to \left(y_1, \pm\infty\right)$ for $y_1 z \neq 0$. Thus, the successive approximations for $\Phi(\mathbf{y}, z)$, as defined by (3) or (6), do not converge globally; however, this does not contradict the fact, established in the proof of the Hartman-Grobman Theorem, that the successive approximations for $\Phi(\mathbf{y}, z)$, as defined by (6), converge locally. It is simply more difficult in this case to determine the function $\Phi(\mathbf{y}, z)$ to which they converge in a neighborhood of the origin. (This is similar to the fact that it is easier to show that $\Sigma\ 1/k^2$ converges than to determine the number to which it converges.)

PROBLEM SET 2.9

3. $\frac{1}{2}\dot{V}(\mathbf{x}) = -x_1^2x_2^2 - x_1^4 - x_2^4 - x_3^2x_1^2 - x_3^4 < 0$ for $\mathbf{x} \neq \mathbf{0}$; so $\mathbf{0}$ is asymptotically stable. The solution of the linearized system $\dot{\mathbf{x}} = D\mathbf{f}(\mathbf{0})\mathbf{x}$ is given by $x_1(t) = x_{10}\cos t - x_{20}\sin t$, $x_2(t) = x_{10}\sin t + x_{20}\cos t$, $x_3(t) = x_{30}$; therefore, the origin of the linearized system is stable, but not asymptotically stable.

5. (b) As noted in the original solutions manual, the easiest way to show that the origin is a saddle for this problem is to compute the eigenvalues of the linear part, $\lambda = 1 \pm \sqrt{3}$, and to use the Hartman-Grobman Theorem. In order to use the Liapunov type function $V(\mathbf{x}) = x_1^2 + x_2^2$ we can use Theorem 3 in Section 3.10, along with the fact that on any given straight line $x_2 = mx_1$ with $|m-2| < \sqrt{3}$, $\dot{V}(\mathbf{x}) < 0$ for all sufficiently small $|\mathbf{x}| \neq 0$ and that on any given straight line $x_2 = mx_1$ with $|m-2| > \sqrt{3}$, $\dot{V}(\mathbf{x}) > 0$ for all sufficiently small $|\mathbf{x}| \neq 0$ (as was noted in the original solutions manual); also, for all sufficiently small $x_1 > 0$ and $-x_1/\sqrt{3} < x_2 < x_1/\sqrt{3}$, $\dot{\theta} < 0$ and $\dot{\theta} \geq 0$ otherwise. This shows that for $x_1 > 0$ there is a separatrix approaching the origin as $t \to \infty$, tangent to the line $x_2 = x_1/\sqrt{3}$, which lies above that line, and a separatrix which approaches the origin as $t \to -\infty$, tangent to the line $x_2 = -x_1/\sqrt{3}$, which lies above that line. Similar results hold for $x_1 < 0$ and there are no other trajectories approaching the origin as $t \to \pm\infty$ for $\mathbf{x} \neq \mathbf{0}$. These facts then imply that the origin is a saddle and is unstable.

6. Since A(t) is continuous, it is integrable and then by direct substitution into the differential equation it follows that

$$\mathbf{x}(t) = \mathbf{x}(0)\exp\int_0^t A(s)ds$$

(with the exponent defined as in Definition 2 in Section 1.3). Thus, by the usual properties of norms, for $B(t) = \int_0^t A(s)ds$, we have

$$\begin{aligned} |\mathbf{x}(t)| &\leq |\mathbf{x}(0)|\,\|I + B(t) + B^2(t)/2! + \ldots\| \\ &\leq |\mathbf{x}(0)|\left[\|I\| + \|B(t)\| + \|B(t)\|^2/2! + \ldots\right] \end{aligned}$$

$$= |\mathbf{x}(0)| \exp\|B(t)\| \le |\mathbf{x}(0)| \exp\int_0^t \|A(s)\| ds.$$

And if $\int_0^\infty \|A(s)\| ds < \infty$, it follows that $\lim_{t\to\infty} |\mathbf{x}(t)| \le |\mathbf{x}(0)| \exp\int_0^\infty \|A(s)\| ds < \infty$.

Problem Set 2.10

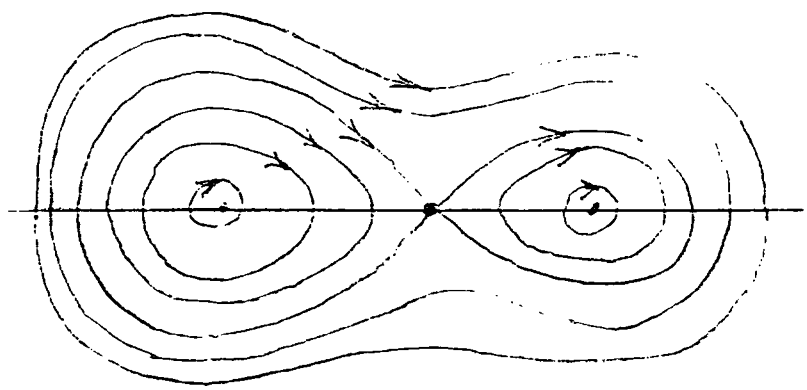
2. By definition of the limit, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{\ln|x|} = 0$. And, by definition,

$f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{1}{\ln|x|} = 0$. Also, for $x \ne 0$, $f'(x) = (\ln|x|-1)/(\ln|x|)^2$ which is continuous for $x \ne 0$ and $\lim_{x\to 0} f'(x) = f'(0) = 0$. Thus, $f \in C^1(R)$. But

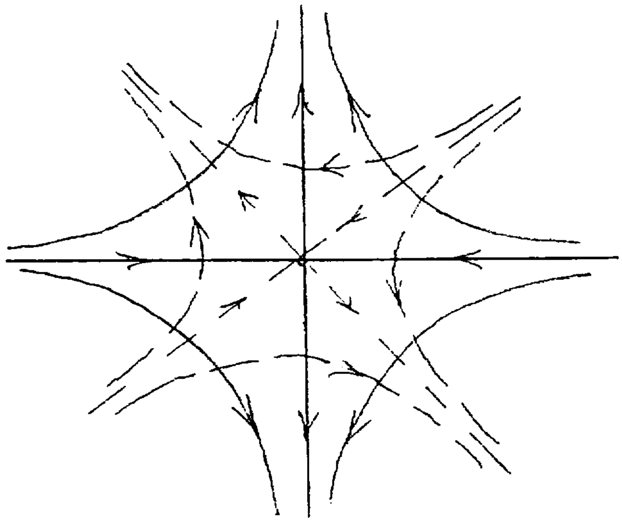
$f''(0) = \lim_{x\to 0} \frac{f'(x)}{x} = \lim_{x\to 0} \frac{(\ln|x|-1)}{x(\ln|x|)^2} = \pm\infty$ does not exist; i.e., $f''(0)$ is undefined.

Problem Set 2.14

4. The phase portrait is given below; cf. Example 1 and Figure 3 in Section 4.9 where $U(x) = -x^2/2 + x^4/4$.



5. (b) Both the Hamiltonian and the gradient systems have saddles at the origin. (The Hamiltonian system phase portrait is shown as dashed curves.)



(d) Both the Hamiltonian and the gradient systems have saddles at (1, 2).

(f) Both the Hamiltonian and the gradient systems have saddles at (−1, 0).

8. $V_x(x, y) = 4x^3 - 6x^2 + 2x = 0$ at $x = 0$, 1 or 1/2 and $V_y(x, y) = 2y = 0$ at $y = 0$. The critical points are at (0, 0), (1, 0) and (1/2, 0). The discriminant $D = V_{xx}V_{yy} - V_{xy}^2$ satisfies $D(1/2, 0) = -1$ and therefore (1/2, 0) is a saddle point; $D(0, 0) = D(1, 0) = 4$ and $V_{yy} = 2 > 0$ and therefore (0, 0) and (1, 0) are local minima. Theorem 5 implies that the gradient system has a saddle at (1/2, 0) and stable nodes at (0, 0) and (1, 0).

10. The system orthogonal to the system in this problem is linearly equivalent to

$$\dot{x} = bx + ay + \text{higher degree terms}$$

$$\dot{y} = -ax + by + \text{higher degree terms}$$

with $a < 0$ and $b > 0$. In the notation of the theorem in Section 1.5 we have $\delta = a^2 + b^2 > 0$, $\tau = 2b > 0$ and $\tau^2 - 4\delta = -4a^2 < 0$. Therefore, according to the theorem in Section 1.5 and Theorem 4 (and the Remark) in Section 2.10, we see that the origin is an unstable focus with a counterclockwise flow. Similarly if (5) has a nondegenerate critical point at the origin which is an unstable focus (i.e. $a > 0$) with a clockwise flow (i.e. $b < 0$) then the system (6) orthogonal to (5) has a stable focus with a clockwise flow, etc.

3. NONLINEAR SYSTEMS: GLOBAL THEORY

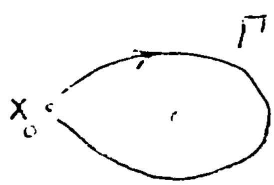
PROBLEM SET 3.1

5. Exactly the same argument, used in the proof of Theorem 1, based on Corollary 2 in Section 2.4, with $\mathbf{f}(\mathbf{x}) / \left(1+|\mathbf{f}(\mathbf{x})|^2\right)$ in place of $\mathbf{f}(\mathbf{x}) / \left(1+|\mathbf{f}(\mathbf{x})|\right)$, can be used to establish a result analogous to Theorem 1 for the differential equation in this problem.

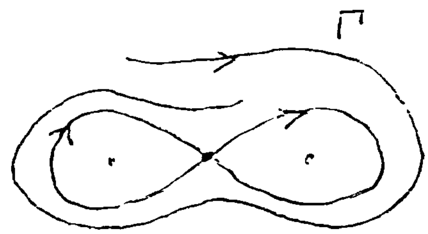
8. As in Problem 7 (with $\tau(\mathbf{x}, t) = t$) we find that for $A = DH(\mathbf{x}_0)$, $A\, Df(\mathbf{x}_0)\, A^{-1} = Dg(H(\mathbf{x}_0))$; i.e., the matrices $Df(\mathbf{x}_0)$ and $Dg(H(\mathbf{x}_0))$ are linearly equivalent and therefore have the same eigenvalues. Note that if $\mathbf{x}_0$ is an equilibrium point of (1), then $H(\mathbf{x}_0)$ is an equilibrium point of (2). Also note that it follows by differentiating $H \circ H^{-1}(\mathbf{x}) = \mathbf{x}$ that $DH(\mathbf{x}_0)\, DH^{-1}(H(\mathbf{x}_0)) = I$; i.e., the matrix $A = DH(\mathbf{x}_0)$ is nonsingular.

PROBLEM SET 3.2

4. (a)

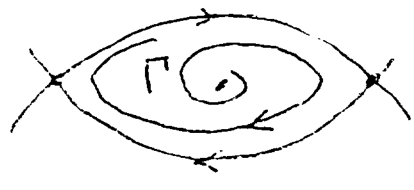


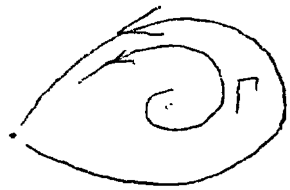
(d)

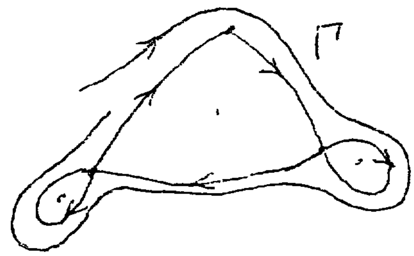


(b)

(c)



(c) 

(f) 

5. (a) In Figure 1 in Section 1.1, the saddle separatrices are invariant, but they are not the α or ω limit sets of any trjaectory of that flow.

(b) The cylinder, A, in Example3 (i.e., in Figure 4) is an attracting set, but it is not the ω-limit set of any trajectory in a neighborhood of A. Also, in Problem 1, the interval $[-1, 1]$ is an attracting set, but it is not the ω-limit set of any trajectory in a neighborhood of $[-1, 1]$.

(c) The cylinder in Example 3 is an attracting set, but it is not an attractor since it does not contain a dense orbit.

PROBLEM SET 3.3

6. The origin is a saddle and there are centers at $(\pm 1, 0)$. The compound separatrix cycle is given by $y^2 - x^2 + x^4/2 = 0$.

PROBLEM SET 3.5

3. Since $\Phi(t)$ satisfies $\dot{\Phi} = A(t)\Phi$ and $\Phi(0) = I$, it follows that $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ satisfies $\dot{\mathbf{x}}(t) = \dot{\Phi}(t)\mathbf{x}_0 = A(t)\Phi(t)\mathbf{x}_0 = A(t)\mathbf{x}(t)$ and $\mathbf{x}(0) = I\,\mathbf{x}_0 = \mathbf{x}_0$.

5. (a) Let $\Phi(t)$ be the fundamental matrix for (2) with $\Phi(0) = I$ and let $\gamma(t)$ be a periodic solution of (1) of period T (where A(t) is a continuous T-periodic matrix). By Theorem 1, $\Phi(T) = Q(T)e^{BT} = Q(0)e^{BT} = e^{BT}$ since $Q(0) = \Phi(0) = I$. Thus, the characteristic multipliers, of $\gamma(t)$, $m_j = e^{\lambda_j T}$, $j = 1, \cdots, n$, are the eigenvalues of $\Phi(T)$ and since the product of the eigenvalues of $\Phi(T)$ is equal to det $\Phi(T)$, it follows from Liouville's Theorem that

$$\prod_{j=1}^{n} m_j = \det\Phi(T) = \exp\int_0^T \operatorname{tr} A(t)dt.$$

And since the sum of the eigenvalues of $\Phi(T)$ is equal to tr $\Phi(T)$, it follows that
$$\sum_{j=1}^{n} m_j = \operatorname{tr}\Phi(T).$$

(b) For $m_1 = e^{\lambda_1 T}$ and $m_2 = 1$ we have from 5(a) that $e^{\lambda_1 T} = \exp\int_0^T \operatorname{tr}A(t)dt$ or that $\lambda_1 = \frac{1}{T}\int_0^T \operatorname{tr} A(t)dt$. But $A(t) = Df(\gamma(t))$ which implies that tr A = tr Df = $\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n} = \nabla\cdot\mathbf{f}$ and therefore $\lambda_1 = \frac{1}{T}\int_0^T \nabla\cdot\mathbf{f}(\gamma(t))dt$. Finally, since $m_1 + m_2 = \operatorname{tr}\Phi(T)$, it follows that $1 + \exp\int_0^T \nabla\cdot\mathbf{f}(\gamma(t))dt = \operatorname{tr}\Phi(T)$.

6. Since $H(t, x_0) = \Phi(t)$ it follows from Liouville's Theorem and $\operatorname{tr} A(t) = \nabla\cdot\mathbf{f}(\gamma(t))$ (Cf. 5(b)) that $\det H(t, x_0) = \det\Phi(t) = \exp\int_0^t \operatorname{tr} A(s)ds = \exp\int_0^t \nabla\cdot\mathbf{f}(\gamma(s))ds$.

PROBLEM SET 3.6

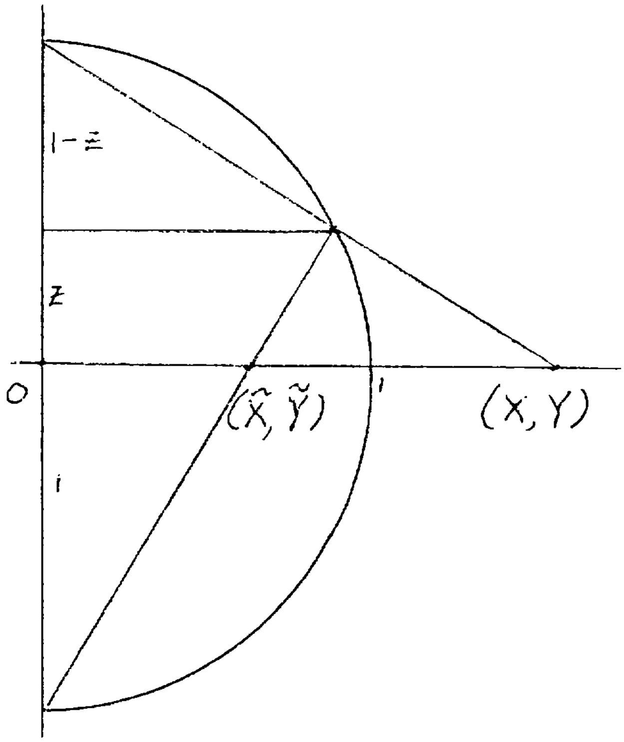
3. This is most easily done by showing that

$$Y = \frac{-\sqrt{k^2-4}\sin t}{D},\ Z = \frac{\pi}{D} \text{ and } W = 0,$$

where $D = k - \sqrt{k^2-4}\cos t$, satisfy the equation of the ellipse (by substituting these quantities into that equation). It can also be accomplished by substituting $x = k - \pi/Z$ and $y = \pi Y/Z$ into $x^2 + y^2 = k^2 - 4$.

5. Under the projective transformation in Problem 3, Γ_0 gets mapped onto the Y-axis; the periodic orbits $\Gamma_\pm: \gamma_\pm(t) = \left(\sqrt{k^2+1/2}\cos t, -\sqrt{k^2+1/2}\sin t, \pm 1, 0\right)$ respectively get mapped onto those branches of the hyperbolas $(Z \pm 2k)^2 - 2Y^2 = 2(2k^2+1)$ which do not intersect the Y-axis (and which are "connected at infinity"). This can be seen by substituting $x = k \mp 1/Z$ and
$y = Y/Z$ into $x^2 + y^2 = k^2 + 1/2$ or alternatively by showing that $Y = -\sqrt{k^2+1/2}\sin t / D$ and $Z = \pm 1/D$, with $D = k - \sqrt{k^2+1/2}\cos t$, satisfy the above equations for the hyperbolas in the (Y, Z) plane. The linearization about Γ_0 shows that Γ_0 has characteristic exponents $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$ and $\lambda_4 = -1$ and that $\dim W^s(\Gamma_0) = \dim W^u(\Gamma_0) = 2$. The linearization about $\Gamma_\pm$ shows that $\Gamma_\pm$ have four zero characteristic exponents; i.e., $\dim W^c(\Gamma_\pm) = 4$.

6. The statement of this problem should have $U_1 = S^2 \sim \{(0,0,1)\}$ and $U_2 = S^2 \sim \{(0,0,-1)\}$. Then for $(x, y, z) \in U_1$, let $\mathbf{h}_1(x, y, z) = (X, Y)$ and for $(x, y, z) \in U_2$, let $\mathbf{h}_2(x, y, z) = (\tilde{X}, \tilde{Y})$, where $\mathbf{h}_1(x, y, z)$ and $\mathbf{h}_2(x, y, z)$ are defined in the statement of this problem. It then follows from the similar triangles in the figure shown here that $(X / Y) = (\tilde{X} / \tilde{Y})$ and that $X^2 + Y^2 = (\tilde{X}^2 + \tilde{Y}^2)^{-1}$. Therefore $\mathbf{h}_2 \circ \mathbf{h}_1^{-1}(X, Y) = (X, Y)/(X^2 + Y^2)$ and for $(X, Y) \in \mathbf{h}_1(U_1 \cap U_2) = \{(X, Y) \in \mathbf{R}^2 | X^2 + Y^2 \neq 0\}$, $D\,\mathbf{h}_2 \circ \mathbf{h}_1^{-1}(X, Y) = [Y^2 - X^2, -2XY; -2XY, X^2 - Y^2]/(X^2 + Y^2)^2$ and $\det D\,\mathbf{h}_2 \circ \mathbf{h}_1^{-1}(X, Y) = -1/(X^2 + Y^2)^2 \neq 0$.



PROBLEM SET 3.7

3. (a) Since $\mathbf{f}$ has no zeros in $\bar{A}$, there are no critical points in $\bar{A}$ and since $\mathbf{f}$ is transverse to the boundary of A, pointing inward, the ω-limit set of any trajectory Γ starting in A is in A. Therefore, by the Poincaré-Bendixson Theorem, ω(Γ) is a periodic orbit which is contained in A.

(b) If A contains a finite number of limit cycles, $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$, ordered such that $\Gamma_j \subset \operatorname{Int} \Gamma_{j+1}$ for $j = 1, \cdots, n-1$, then Γ_1 must be stable on its interior since it is the ω-limit set of any trajectory starting in $A \cap \operatorname{Int} \Gamma_1$. Similarly, Γ_n is stable on its exterior. If Γ_1 is a stable limit cycle, we are done. If Γ_1 is not a stable limit cycle, then it must be a semi-stable limit cycle, unstable on its exterior; and then Γ_2 must be stable on its interior since it is the ω-limit set of any trajectory starting in Ext $\Gamma_1 \cap$ Int Γ_2. Continuing in this way, we find that either there exists an integer j with $1 < j < n$ such that Γ_j is stable or that Γ_n is stable on its interior; i.e., that Γ_n is a stable limit cycle. In either case there exists at least one stable limit cycle in A.

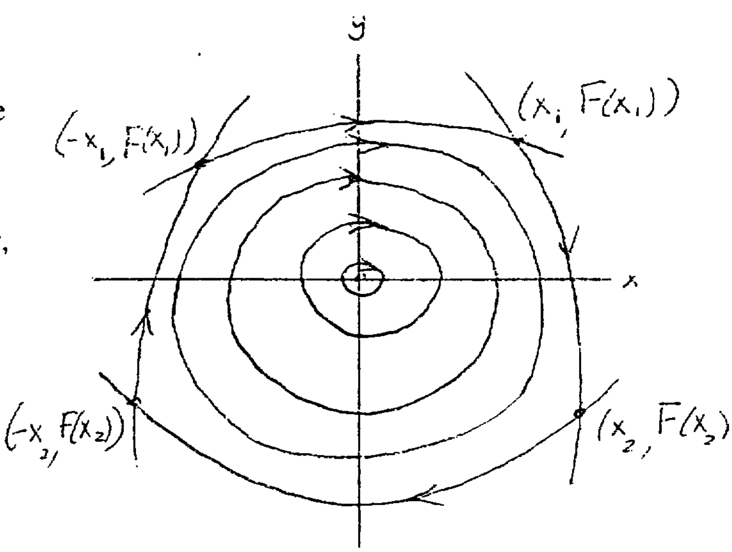
5. The only critical point is at the origin. $r\dot{r} = y^2(1 - x^2 - y^2) = 0$ on $r = 1$ and $\dot{\theta} < 0$ on $r = 1$. Also, $\dot{r} < 0$ for $r > 1$ and $y \neq 0$ while $\dot{r} > 0$ for $0 < r < 1$ and $y \neq 0$. Thus, $r = 1$ is a stable limit cycle which is the ω-limit set of every trajectory in $\mathbf{R}^2 \backslash \{\mathbf{0}\}$.

PROBLEM SET 3.8

1. Clearly F, g $\in C^1(\mathbf{R})$, F and g are odd functions, $xg(x) = x^2 > 0$ for $x \neq 0$, $F(0) = 0$, $F'(0) = (x^4 + 4x^2 - 1)/(x^2 + 1)^2\big|_{x=0} = -1 < 0$, F has a single positive zero at $x = 1$ and since $F'(x) > 0$ for $x > \sqrt{5} - 2$ (where $\sqrt{5} - 2 < 1$), $F(x) \to \infty$, monotonically for $x \geq 1$, as $x \to \infty$. Thus, F and g satisfy the hypotheses of Lienard's Theorem.

5. (a) There is a center at the origin and the phase portrait is topologically equivalent to Figure 4 in Section 1.5 (with $b < 0$).

(b) Assuming that $F(0) = 0$, it follows from Theorem 6 in Section 2.10 that the origin is a center for this system since it is symmetric with respect to the y-axis. (In any case, there is a center at $(0, F(0))$ according to Theorem 6.) If $g(x)$ has no zeros, other than $x = 0$, then the phase portrait is topologically equivalent to Figure 4 in Section 1.5 (with $b < 0$); however, if for example $g(x)$ has zeros at $\pm x_1$ and $\pm x_2$ (where $0 < x_1 < x_2$) then, according to Theorem 3 in [24], the outer boundary of the continuous band of cycles around the center at the origin is a graphic (defined in Section



3.7) and we could have the phase portrait shown here. It could also happen that the graphic includes the "point at infinity" on the Bendixson sphere, shown in Figure 1 in Section 3.6, in which case part of the continuous band of cycles would extend to infinity.

7. If we let $x = \dot{z}$ and $y = -z$, then the second-order differential equation

$$\ddot{z} + F(\dot{z}) + z = 0 \qquad (*)$$

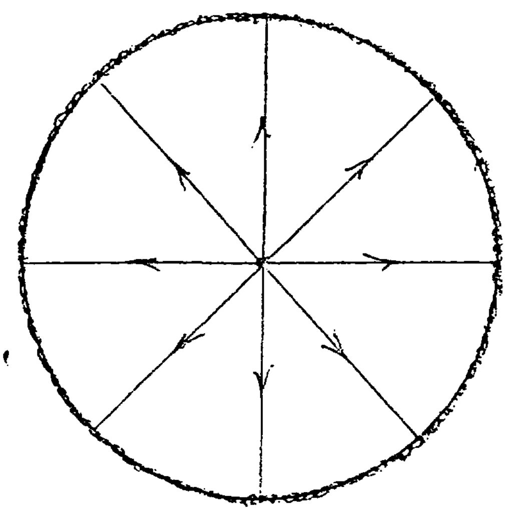
can be written in the form of the Lienard system (1) with $g(x) = x$. Thus, if $F(x)$ satisfies the hypotheses of Lienard's Theorem (and $g(x) = x$), it follows from Lienard's Theorem that (1) has a unique stable limit cycle, i.e., (*) has a unique, asymptotically stable, periodic solution.

Problem Set 3.9

5. (a) $\nabla \cdot (B\mathbf{f}) = -b^2 e^{-2\beta x} < 0$ which implies that this system has no limit cycle in $\mathbf{R}^2$, by Theorem 2.

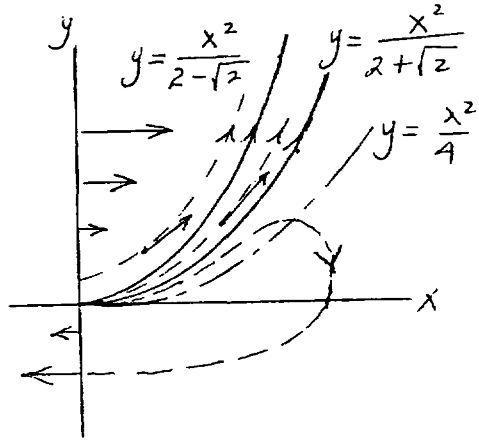
Problem Set 3.10

3. (b) The projection of the flow on the (y, z) and (x, z) planes, according to Theorem 2, is given by $\dot{y} = 0$, $\dot{z} = -z$ and $\dot{x} = 0$, $\dot{z} = -z$ respectively; and the global phase portrait is given by the following figure which has an unstable proper node at the origin and a circle of critical points at infinity.



PROBLEM SET 3.11

5. First of all, the functions $y = x^2/\left(2 \pm \sqrt{2}\right)$ satisfy the differential equations of this problem in the form $dy/dx = \left(-x^3 + 4xy\right)/y$ since both sides reduce to $2x/\left(2 \pm \sqrt{2}\right)$ for $y = x^2/\left(2 \pm \sqrt{2}\right)$. Next, since the system in this problem is invariant under the transformation $(t, x, y) \to (-t, -x, y)$, it is symmetric with respect to the y-axis. And then for $x = 0$ we have $\dot{y} = 0$ and $\dot{x} > 0$ for $y > 0$ while $\dot{x} < 0$ for $y < 0$; and $\dot{y} > 0$ above the parabola $y = x^2/4$ for $x > 0$; and both $\dot{x} < 0$ and $\dot{y} < 0$ for $x > 0$ and $y < 0$. Thus, we have the vector field and flow shown here. This, combined with the symmetry, shows that there is a



hyperbolic sector above the parabola $y = x^2/\left(2 - \sqrt{2}\right)$, a parabolic sector between the parabolas $y = x^2/\left(2 \pm \sqrt{2}\right)$ and an elliptic sector below the parabola $y = x^2/\left(2 + \sqrt{2}\right)$. Using equation (7′) in Theorem 2 in Section 3.10 we find that the projection of the flow on the Poincaré sphere onto the (x, z) plane at the point (0, 1, 0) satisfies $\dot{x} = z^2 - 4x^2z + x^4$ and $\dot{z} = -4xz^2 + x^3z$. This system is symmetric about the z-axis and $z = x^2/\left(2 \pm \sqrt{2}\right)$ are invariant curves of this system in the form $dz/dx = \left(-4xz^2 + x^3z\right)/\left(z^2 - 4x^2z + x^4\right)$ since both sides reduce to $2x/\left(2 \pm \sqrt{2}\right)$ for $z = x^2/\left(2 \pm \sqrt{2}\right)$. An analysis similar to that given above then allows us to complete the description of the types of sectors that this system has at the origin, as listed in the Hint for this problem. We thus obtain the separatrix configuration shown in this problem (on p. 398) and we see that, according to Definition 1, the four trajectories which lie on the invariant parabolas $y = x^2/\left(2 \pm \sqrt{2}\right)$, which correspond to the invariant parabolas $z = x^2/\left(2 \pm \sqrt{2}\right)$ of equation (7′), as given above, for $x \neq 0$ are separatrices since $y = x^2/\left(2 - \sqrt{2}\right)$ lies on the boundary of the hyperbolic sector above this parabola through the origin, and similarly, $y = x^2/\left(2 + \sqrt{2}\right)$ lies on the boundary of the two hyperbolic sectors between this parabola and the equator of the Poincaré sphere at the point (0, 1, 0) at infinity.

PROBLEM SET 3.12

2. With the coordinate system and the Jordan curve C_α as described in this problem, it follows from the uniform continuity of $\mathbf{f}(\mathbf{x})$ that for $k_0 > 0$, there exists a $\delta > 0$ such that if C_α is contained in a square of side δ in E (as in the proof of Theorem 1), then for all $\mathbf{x} \in C_\alpha$, $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < k_0/2$. This implies that for all $\mathbf{x} = (x, y)^T \in C_\alpha$, $|Q(x,y)| < k_0/2$ and $|P(x,y) - k_0| < k_0/2$; and therefore that $|P(x,y)| > k_0/2$, i.e., that $|Q(x,y)/P(x,y)| < 1$. It follows that $-\pi/4 < \tan^{-1}|Q(x,y)/P(x,y)| < \pi/4$ for all $(x, y)^T \in C_a$. Thus, as (x, y) moves around C_α in the positive direction, $\Delta\,\Theta < \pi/2$ or $\Delta\,\Theta/2\pi < 1/4$.

4. NONLINEAR SYSTEMS: BIFURCATION THEORY

PROBLEM SET 4.1

2. (a) As in the solution of Problem 1(a), since $\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x}) = -\mu\mathbf{x}$, we have

$$\|\mathbf{f} - \mathbf{g}\|_1 = |\mu|\left(\max_{x\in K}|x| + 1\right).$$

(b) If there were a homeomorphism $H : \mathbf{R}^2 \to \mathbf{R}^2$ and a strictly increasing function $\tau : \mathbf{R} \to \mathbf{R}$ such that $H \circ \varphi_t = \psi_\tau \circ H$, then for $\mathbf{x} \in \mathbf{R}^2$ we would have $\lim_{t\to\infty} |H \circ \varphi_t(\mathbf{x})| = \lim_{t\to\infty} |\psi_t \circ H(\mathbf{x})|$. But for $\mu > 0$ and $0 < |\mathbf{x}| < 1$ we have $|\varphi_t(\mathbf{x})| \le 1$ for all $t \ge 0$ and thus there is a constant M such that $|H \circ \varphi_t(\mathbf{x})| \le M$ for all $t \ge 0$ (since a continuous function on a compact set is bounded); and $\lim_{t\to\infty} |\psi_t \circ H(\mathbf{x})| = \infty$ (since for $\mathbf{x} \ne \mathbf{0}$, $H(\mathbf{x}) \ne 0$; cf. Figure 2). We therefore have a contradiction. Similarly, for $\mu > 0$ and $1 < |\mathbf{x}| < 1 + \varepsilon$, with $\varepsilon > 0$ sufficiently small, we have $|H \circ \varphi_t(\mathbf{x})| \le M$ for all $t \ge 0$ and $\lim_{t\to\infty} |H \circ \psi_t(\mathbf{x})| = \infty$ and we again arrive at a contradiction. Thus the two systems in Example 2 are not topologically equivalent for $\mu \ne 0$.

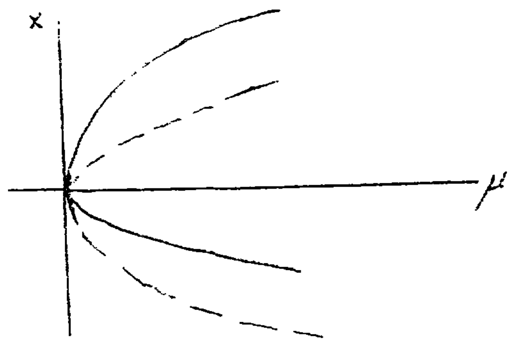
9. According to the solution in Problem 10 in Section 3.10, 10 (a, b, c, d, e) correspond to the global phase portraits in Figure 12 (i, vii, v, vi, ii). Thus, we see that for 10 (a, b, e) the nonwandering set on the Poincaré sphere, S^2, is simply the set of critical points on S^2

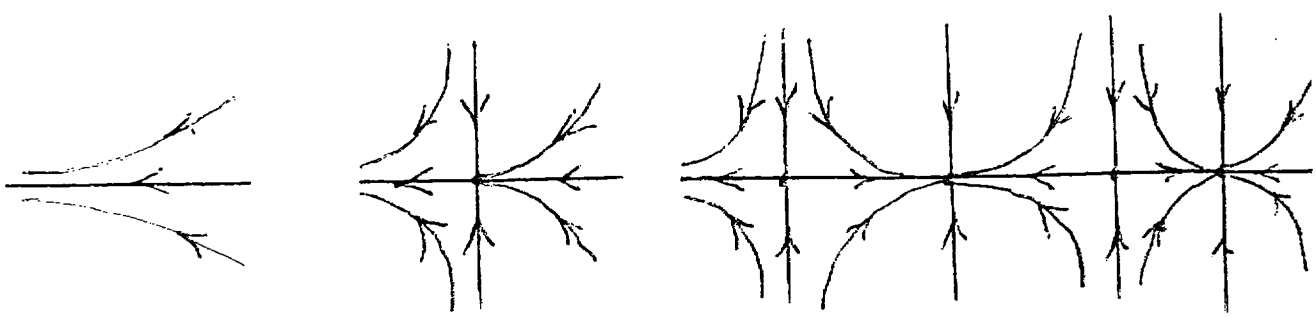
as shown in Figure 12 (i, vii, ii) in Section 3.10. For 10 (c), the nonwandering set consists of the set of critical points on S^2 and the separatrix cycle or graphic consisting of the homoclinic loop at the origin and the origin shown in Figure 12 (v). For 10 (d) the nonwandering set consists of the set of critical points on S^2 and the two graphics consisting of the hetroclinic loops from (1, 0) to (–1, 0), these two critical points and the piece of the x-axis between these two critical points shown in Figure 12 (vi).

Problem Set 4.2

6. Let the hyperbolic critical points that occur near $\mathbf{x}_0$ for $\mu > \mu_0$ be $\mathbf{x}_\pm$. We then have dim $W^s(\mathbf{x}_+) = k + 1$, dim $W^u(\mathbf{x}_+) = n - k - 1$, dim $W^s(\mathbf{x}_-) = k$ and dim $W^u(\mathbf{x}_-) = n - k$. If the conditions (3) are satisfied, then there are two hyperbolic critical points near $\mathbf{x}_0$ for both $\mu > \mu_0$ and $\mu < \mu_0$ and the dimensions of the stable and unstable manifolds are the same as above. If the conditions (4) are satisfied then there are three hyperbolic critical points near $\mathbf{x}_0$ for $\mu > \mu_0$ (or for $\mu < \mu_0$) and the dimensions of the stable and unstable manifolds are the same as those above for $\mathbf{x}_-$ at two of the critical points and the same as those above for $\mathbf{x}_+$ at the remaining critical point.

7. The critical points are at $\left(\pm 2\sqrt{\mu}, 0\right)$ and $\left(\pm\sqrt{\mu}, 0\right)$ for $\mu \geq 0$. The bifurcation diagram is shown at the right and the phase portraits are shown below.





$\mu < 0$ $\mu = 0$ $\mu > 0$

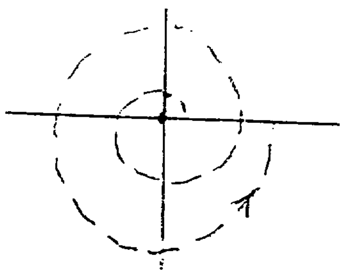
Problem Set 4.3

5. A universal unfolding for the system in this problem is given by $\dot{x} = \mu_1 + \mu_2 x + \mu_3 x^2 - x^4$, $\dot{y} = -y$; and the various phase portraits for (μ_1, μ_2, μ_3) in the parameter space shown in Figure 3 can be determined from those in Problem 1.

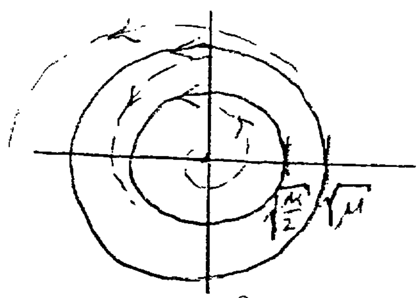
Problem Set 4.4

1. (b) Writing the system in 1(b), without any $O(r^4)$ terms, in polar coordinates leads to $\dot{r} = r(\mu + ar^2)$ and $\dot{\theta} = 1$. Thus, for $\mu = 0$, $dr/d\theta = ar^3$ which has the solution $r(0) = r_0\left[1 - 2ar_0^2\theta\right]^{-1/2}$. This implies that the Poincaré map $P(r_0) = r_0\left[1 - 4\pi a r_0^2\right]^{-1/2}$. Thus, $P'(r_0) = \left[1 - 4\pi a r_0^2\right]^{-3/2}$, $P''(r_0) = 12\pi a r_0\left[1 - 4\pi a r_0^2\right]^{-5/2}$ and $P'''(r_0) = 12\pi a\left[1 - 4\pi a r_0^2\right]^{-5/2} + O(r_0)$ as $r_0 \to 0$. This shows that $d(0) = P(0) = 0$, $d'(0) = P'(0) - 1 = 0$, $d''(0) = P''(0) = 0$ and that $d'''(0) = P'''(0) = 12\pi a = \sigma$, since from equation (3), $\sigma = 12\pi a$.

4. $\dot{r} = r(\mu - r^2)(\mu - 2r^2)$, $\dot{\vartheta} = 1$. Therefore, for $\mu > 0$, we have $\dot{r} = 0$ for $r = 0$, $r = \sqrt{\mu}$ and $r = \sqrt{\mu/2}$. On $\gamma_1(t)$, $\dot{x}(t) = -\sqrt{\mu}\,\sin t = -y(t)$ and $\dot{y}(t) = \sqrt{\mu}\cos t = x(t)$; i.e., $\gamma_1(t)$ is a periodic solution of the system in this problem, as is $\gamma_2(t)$. Since $\dot{r} > 0$ for $0 < r < \sqrt{\mu/2}$ and for $r > \sqrt{\mu}$ while $\dot{r} < 0$ for $\sqrt{\mu/2} < r < \sqrt{\mu}$, we see that $\gamma_1(t)$ is an unstable limit cycle and $\gamma_2(t)$ is a stable limit cycle of this system; and the origin is an unstable focus. The phase portraits are given by

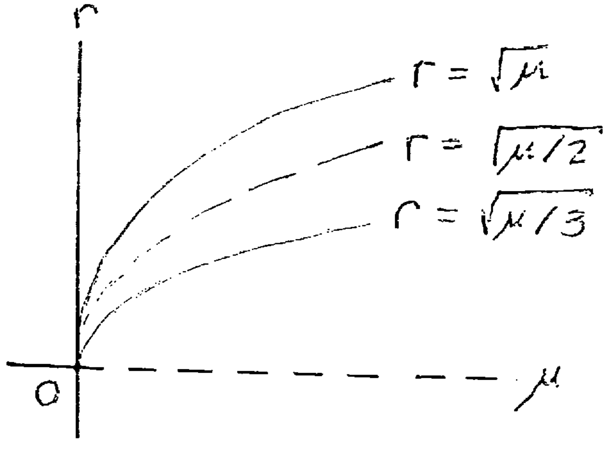


$\mu \le 0$



$\mu > 0$

5. The bifurcation diagram for the system in this problem is shown here.

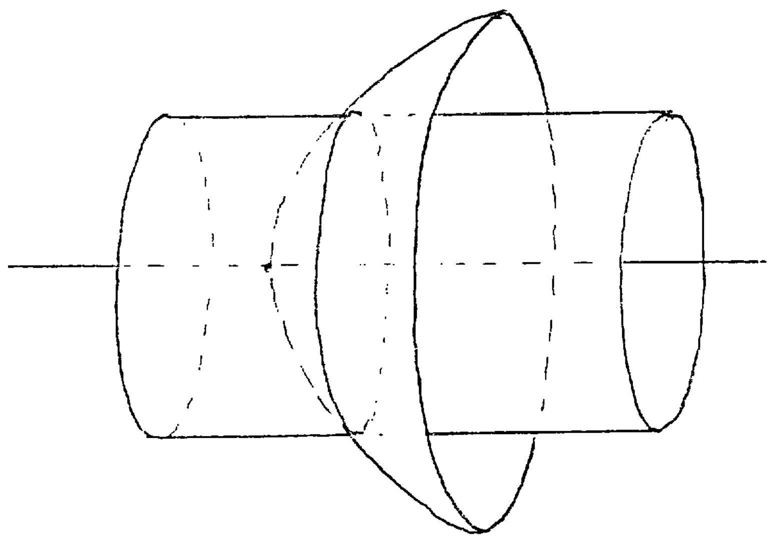


9. Substituting $a = -1$, $b = E$, $c = F$, $d = 1$, $a_{02} = 1$, $b_{11} = -1$ and $b_{02} = c$ into equation (3′) leads directly to the result stated in this problem. (Cf. Problems 1 and 5 in Section 4.14 and note that the division sign should be deleted in the formula for σ in Problems 1 and 5 in the original Solutions Manual.)

PROBLEM SET 4.5

2. Equation (2) in Example 2 yields $DP(\gamma_\mu^\pm, \mu) = \exp\int_0^{2\pi} 4r^2(r^2-1)dt = \exp\left[\pm 8\pi\mu^{1/2}(1+\mu^{1/2})\right]$; in Example 3 it yields $DP(\gamma_0, \mu) = \exp\int_0^{2\pi} 2r^2(1+\mu-r^2)dt = \exp(4\pi\mu)$ and $DP(\gamma_\mu, \mu) = \exp\int_0^{2\pi} 2r^2(1-r^2)dt = \exp[-4\pi\mu(1+\mu)]$; and in Example 4 it yields $DP(\gamma_0, \mu) = \exp = \int_0^{2\pi} -2r^2\left[\mu-(r^2-1)\right]^2 dt = \exp(-4\pi\mu)$ and $DP(\gamma_\mu^\pm, \mu) = \exp\int_0^{2\pi} 4r^2(1-r^2)^2 dt = \exp\left[8\pi\mu(1\pm\mu^{1/2})\right]$.

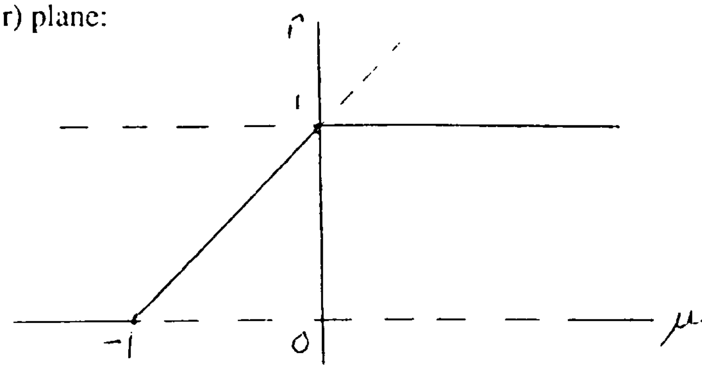
3. The surfaces of periodic orbits in Example 3 are given by:



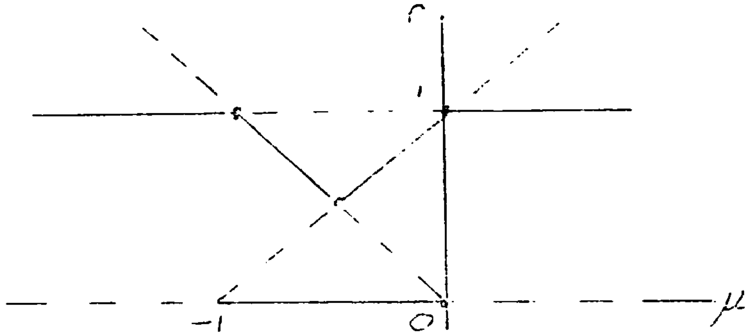
And the surfaces for Example 4 are given by the surface in Figure 3 in Section 4.5 intersected with the unit cylinder along the μ-axis.

6. In polar coordinates, these systems have the form $\dot{r} = r\psi(r, \mu)$, $\dot{\varphi} = 1$. The bifurcation diagram is given by $\psi(r, \mu) = 0$ in the (μ, r) plane:

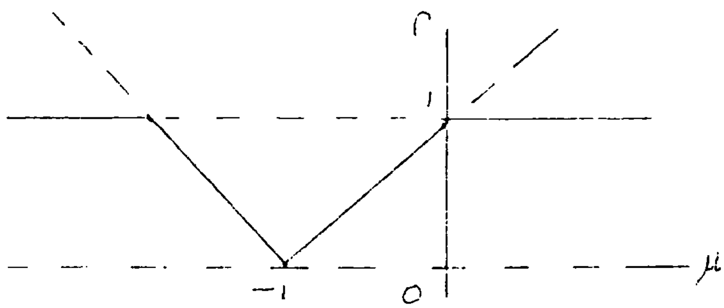
(a) For $\psi(r, \mu) = (r - 1)(r - \mu - 1)$, the bifurcation diagram is given here; there is a Hopf bifurcation at $(-1, 0)$ and a transcritical bifurcation at $(0, 1)$.



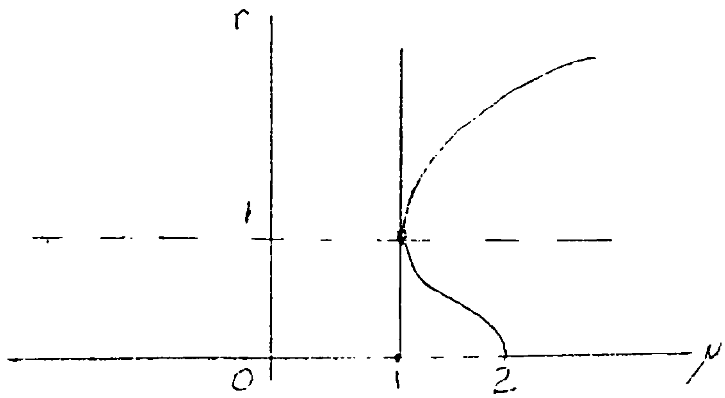
(b) For $\psi(r, \mu) = (r - 1)(r - \mu - 1)(r + \mu)$, the bifurcation diagram is given here; there are Hopf bifurcations at $(-1, 0)$ and at $(0, 0)$ and transcritical bifurcations at $(-1, 1)$, $(-1/2, 1/2)$ and $(0, 1)$.



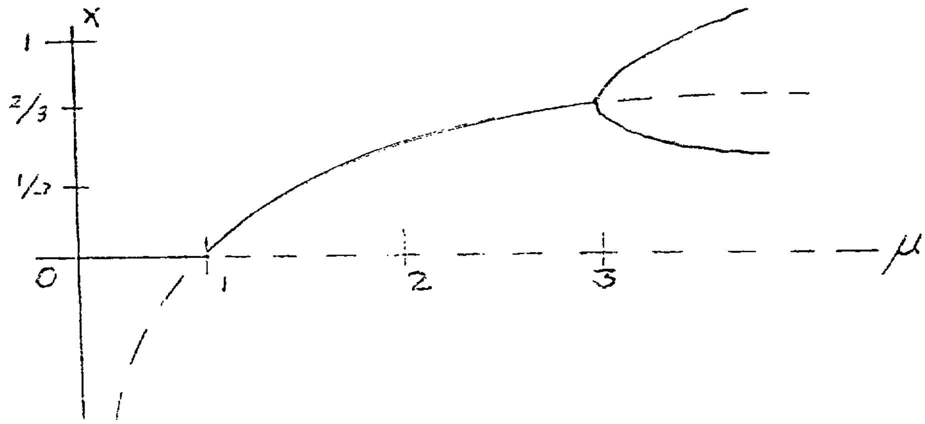
(c) For $\psi(r, \mu) = (r - 1)(r - \mu - 1)(r + \mu + 1)$, the bifurcation diagram is given here; there is a Hopf type bifurcation at $(-1, 0)$ and transcritical bifurcations at $(-2, 1)$ and $(0, 1)$.



(d) For $\psi(r, \mu) = (\mu - 1)(r^2 - 1)\left[\mu - 1 - \left(r^2 - 1\right)^2\right]$, the bifurcation diagram is given here; there is a Hopf bifurcation at $(2, 0)$, a center for $\mu = 1$ and $r \geq 0$, and a higher codimension bifurcation at $(1, 1)$.



10. For $P(x, \mu) = \mu x(1 - x)$, $x = 2/3$ is a fixed point of $P(x, 3) = 3x(1 - x)$ since $P(2/3, 3) = 2/3$; it is nonhyperbolic since $DP(x, \mu) = \mu(1 - 2x)$ implies that $DP(2/3, 3) = -1$. We therefore expect a period doubling, or flip bifurcation. As in Problem 9, we see that this is indeed the case since for $F(x, \mu) \equiv P^2(x, \mu) = \mu^2 x(1 - x)(1 - \mu x + \mu x^2)$ we have $F(2/3, 3) = 2/3$, $DF(x, \mu) = \mu^2(1 - 2x)(1 - 2x + 2\mu x^2)$ which implies that $DF(2/3, 3) = 1$, $D^2F(x, \mu) = -2\mu^2\left[1 - 2\mu x + 2\mu x^2 + \mu(1 - 2x)^2\right]$ which implies that $D^2F(2/3, 3) = 0$, $D^3F(x, \mu) = 8\mu^3(1 - 2x)$ which implies that $D^3F(2/3, 3) = -72 \neq 0$, $F_\mu(x, \mu) = 2\mu x(1 - x)(1 - \mu x + \mu x^2) - \mu^2 x^2(1 - x)^2$ which implies that $F_\mu(2/3, 3) = 0$ and $DF_\mu(x, \mu) = 2\mu(1 - 2x)(1 - 3\mu x + 4\mu x^2)$ which implies that $DF_\mu(2/3, 3) = 2/3 \neq 0$. Thus, conditions (4) in Section 4.2 are satisfied for the map $F = P^2$ and therefore P^2 has a pitchfork bifurcation at $(2/3, 3)$. The bifurcation diagram is obtained by graphing the equation $F(x, \mu) = x$; this was done using Implicit Plot on Mathematica. The resulting bifurcation diagram is shown here.

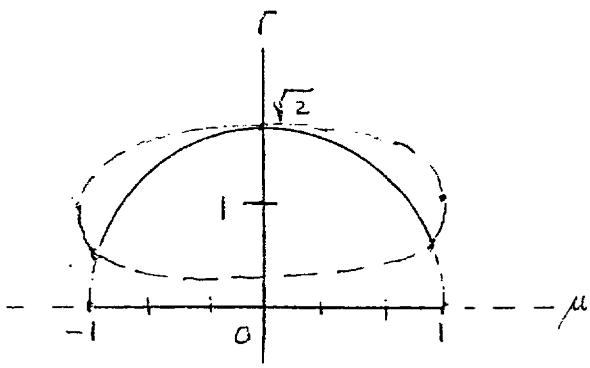


PROBLEM SET 4.7

3. The bifurcation diagram is shown here. As in the original Solutions Manual, it is given by the graph of the relation

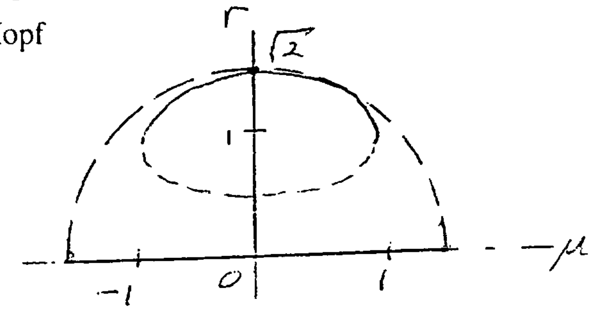
$$\left[(r^2 - 2)^2 + \mu^2 - 1\right]\cdot\left[r^2 + 2\mu^2 - 2\right] = 0$$

in the (μ, r) plane. There are subcritical



Hopf bifurcations at $(\pm 1, 0)$, saddle node bifurcations at the nonhyperbolic periodic orbits corresponding to the points $(\pm 1, 1)$, there are transcritical bifurcations at the periodic orbits corresponding to the intersection points of these two conic sections with $\mu \neq 0$, and there is a higher codimension bifurcation at the periodic orbit corresponding to the point $(0, \sqrt{2})$.

4. The bifurcation diagram is shown here. As in the original Solutions Manual, it is given by the graph of the relation $\left[(r^2-2)^2+\mu^2-1\right]\cdot\left[r^2+\mu^2-3\right]=0$ in the (μ, r) plane. There are subcritical Hopf bifurcations at $\left(\pm\sqrt{3/2}, 0\right)$, saddle node bifurcations at $(\pm 1, 1)$ and a higher codimension bifurcation at $(0, \sqrt{2})$.



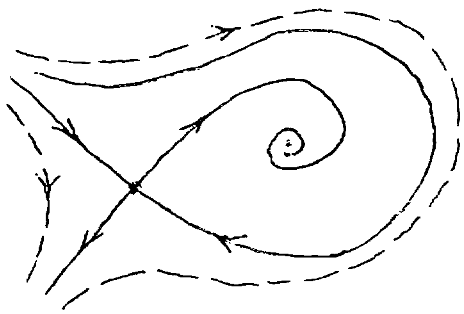
Problem Set 4.8

3. (a) This is a Hamiltonian system with Hamiltonian $H(x, y) = y^2 - 6x^2 + x^3$. There are critical points at (0, 0) and (4, 0), a saddle and a center respectively. Note that this system is symmetric about the x-axis. The phase portrait is topologically equivalent to Figure 1 in Section 4.9. The homoclinic loop Γ_0 is given by $y^2 = 6x^2 - x^3$. And we note that $H(4, 0) = -32$.

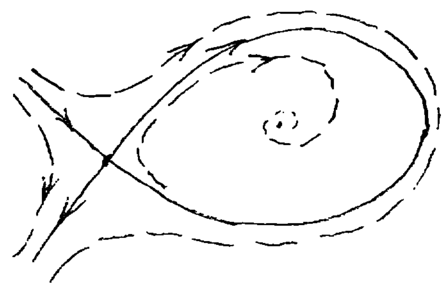
(b) The system $\dot{x} = X(x, y, \alpha) = y - \alpha H(x, y)(12x - 3x^2)$, $\dot{y} = 12x - 3x^2 + \alpha H(x, y)y$ satisfies $[P, Q; P_\alpha, Q_\alpha] = H(x, y)\left[y^2 + (12x - 3x^2)^2\right] < 0$ for $H(x, y) < 0$, i.e., on the interior of Γ_0 (which is an invariant curve of this system for all α). As in 3(b), for $\alpha > 0$ the loop Γ_0 is internally unstable, $\sigma = -1$ and $\omega = -1$. The phase portrait is shown below for $\alpha > 0$ (and $\mu = 0$).

(c) If we fix α at a positive value and embed the vector field (X, Y) of part (b) in a one-parameter family of rotated vector fields (5), then from Figure 1 and Theorem 3 in Section 4.6 an unstable limit cycle bifurcates from the interior of Γ_0 as μ increases from zero. The trace of the linear part of (5) at the critical point (4, 0) is given by $\tau_\mu = \text{trace } Df(4, 0, \mu) = 13[-32\alpha \cos\mu + \sin\mu]$, where $\mathbf{f}(x, y, \mu)$ is the vector field (5). And we have $\tau_\mu = 0$ for $\mu =$

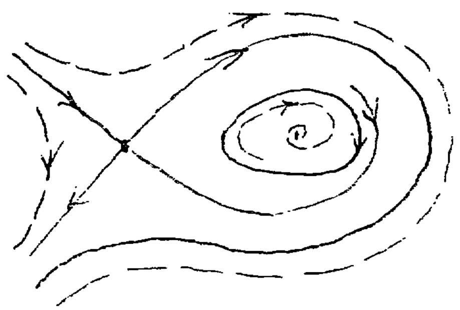
$\mu^* \equiv \tan^{-1}(3.2) \approx 1.27$ for $\alpha = .1$. The phase portraits for the system (5) are described below:



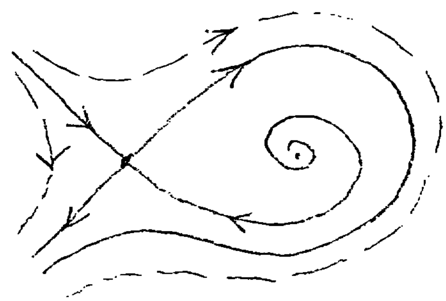
$\mu < 0$



$\mu = 0$



$0 < \mu < \mu^*$



$\mu \geq \mu^*$

(c) For $\alpha = -.1$ the separatrix cycle Γ_0 is internally stable and thus a stable limit cycle bifurcates from the interior of Γ_0 as μ decreases from zero.

PROBLEM SET 4.11

6. For $\gamma_\alpha(t) = (-\alpha\cos t, \alpha\sin t)$, we have $M_1(\alpha) \equiv 0$. And then $f(x, y, \varepsilon) = \varepsilon a_1 x + \varepsilon a_2 x^3 + \varepsilon a_3 x^5 + \cdots + \varepsilon a_m x^{2m-1} + x^{2m}$, $g(x, y, \varepsilon) = A_1 x^{2m} + A_2 x^{2m-2} + \cdots + A_{m-1} x^4 + A_m x^2$, $f_x(x, y, 0) = 2mx^{2m-1}$, $f_\varepsilon(x, y, \varepsilon) = a_1 x + a_2 x^3 + \cdots + a_m x^{2m-1}$, $g_y = 0$, $g_\varepsilon = 0$, $F(x, y) = x^{2m}y - A_1 x^{2m+1}/(2m+1) - A_2 x^{2m-1}/(2m-1) - \cdots - Amx^3/3$, $G(x, y) = 2mx^{2m-1}y$, $G_1(x, y) = 2mx^{2m-1}y$, $G_2 = 0$, $P_2 = 0$, $P_{2h} = 0$ and $G_{1h}(x, y) = 2mx^{2m-1}/y$. Thus, from

Theorem 1 with $dx = ydt$, $dy = -xdt$, to first order, and with $h = \alpha^2/2$ and the integrals being taken around $\gamma_\alpha(t)$, we have

$$M_2(\alpha) = \oint [G_{1h}P_2 - G_1P_{2h}]dx + \oint [g_\varepsilon dx - f_\varepsilon dy] - \oint \frac{F}{y}[f_x + g_y]dx$$

$$= -\oint [a_1x + a_2x^3 + \ldots + a_mx^{2m-1}]dy$$

$$-\oint \left[x^{2m}y - \frac{A_1x^{2m+1}}{2m+1} - \frac{A_2x^{2m-1}}{2m-1} - \ldots - \frac{Amx^3}{3}\right]\left[\frac{2mx^{2m-1}}{y}\right]dx$$

$$= \int_0^{2\pi} [a_1x_\alpha^2(t) + a_2x_\alpha^4(t) + \ldots + a_mx_\alpha^{2m}(t)]dt$$

$$+\int_0^{2\pi} 2m\left[\frac{A_1}{2m+1}x_\alpha^{4m}(t) + \frac{A_2}{2m-1}x_\alpha^{4m-2}(t) + \ldots + \frac{Am}{3}x_\alpha^{2m+2}(t)\right]dt$$

$$= 4m\pi\alpha^2\left[\frac{A_1}{(2m+1)2^{2m}}\binom{4m}{2m}(\alpha^2)^{2m-1} + \frac{A_2}{(2m-1)2^{2m-1}}\binom{2(2m-1)}{2m-1}(\alpha^2)^{2m-2} + \cdots\right.$$

$$\left. + \frac{Am}{3\cdot 2^{m+1}}\binom{2(m+1)}{m+1}(\alpha^2)^m + \frac{a_m}{2^m}\binom{2m}{m}(\alpha^2)^{m-1} + \ldots + \frac{a_1}{2}\right] = \pi\alpha^2 P_{2m-1}(\alpha^2)$$

where $P_{2m-1}(\alpha^2)$ is a $(2m - 1)$th degree polynomial in α^2. Note that the constant

$k = \dfrac{4m}{(2m+1)2^{2m}}\dbinom{4m}{2m} \neq 0$ in Problem 6. We have used the formula

$\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos^{2m} t\, dt = \binom{2m}{m} \cdot \frac{1}{2^{2m}}$ given in Theorem 6 in Section 3.8 in obtaining this result. Thus, we see that for sufficiently small $\varepsilon \neq 0$ the system in Problem 6 has $2m - 1$ limit cycles for an appropriate choice of constants $a_1, \cdots, a_m, A_1, \cdots, A_m$ (alternating in sign). In fact, if we wish to obtain limit cycles asymptotic to circles of radius r_j, $j = 1, \cdots, 2m - 1$, as $\varepsilon \to 0$, we simply set the $(2m - 1)$th degree polynomial $(\alpha^2 - r_1^2)(\alpha^2 - r_2^2) \cdots (\alpha^2 - r_{2m-1}^2)$ in α^2 equal to $k_0P_{2m-1}(\alpha^2)$ with any non-zero constant k_0, in order to determine the $2m$ coefficients $a_1, \cdots, a_m, A_1, \cdots, A_m$.

PROBLEM SET 4.12

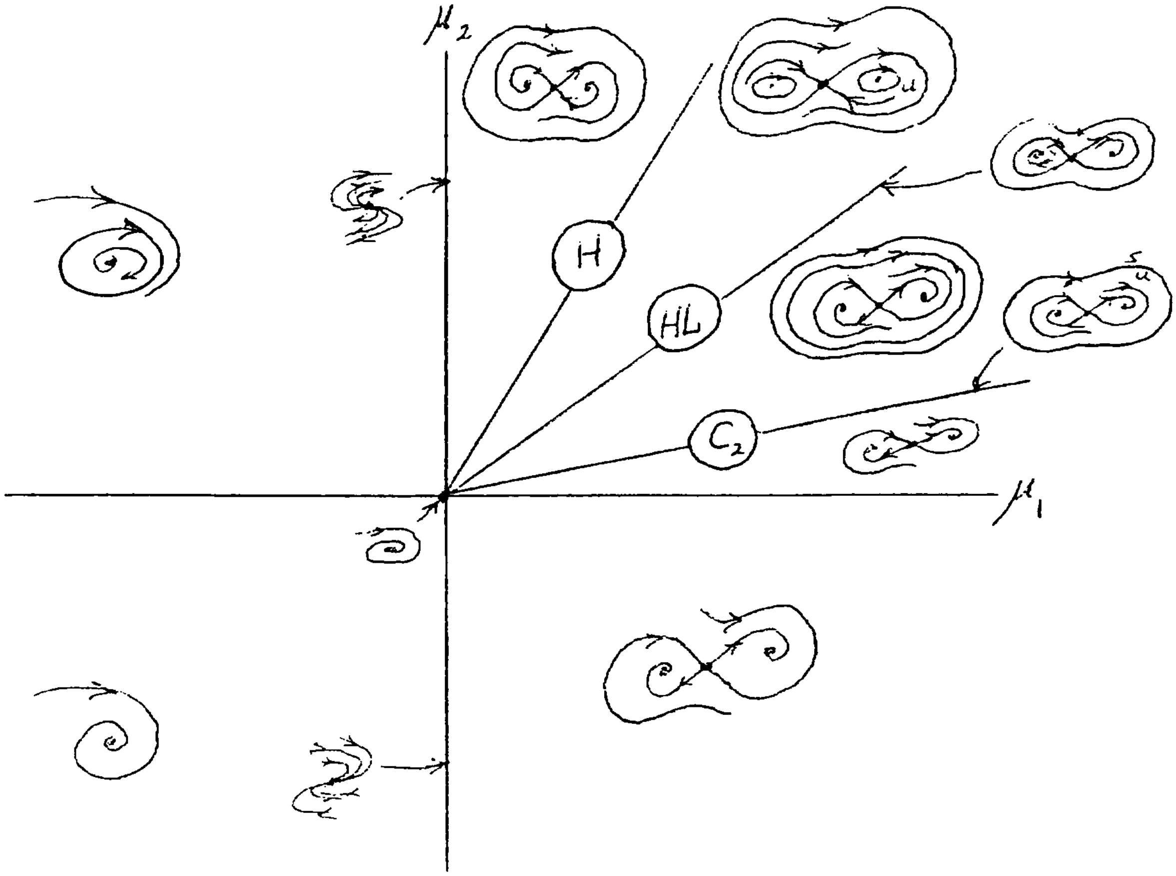
2. (c) All of the results necessary for the computation of $d_4(h)$ are derived in Problem 1(b). For $\lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{51} = \lambda_{52} = 0$, we have $f_1(x, y) = -2x^2 - 2\lambda_{21} xy + y^2$, $g_1(x, y) = -\lambda_{21}x^2 + (4 + \lambda_{41})xy + \lambda_{21}y^2$, $f_2(x, y) = g_2(x, y) = g_3(x, y) = 0$, $f_3(x, y) = -\lambda_{53}xy$, $f_4(x, y) = \lambda_{14}x$ and $g_4(x, y) = \lambda_{14}y$. Then q_1, q_2 and Ω_2 are the same as in part (b), with $\lambda_{52} = 0$, and $\Omega_3 = \lambda_{53}\, xydy + \lambda_{41}x\, \{\lambda_{21}(2 + \lambda_{41})x^3 + [\lambda_{21}(7\lambda_{21} - 4 - \lambda_{41})/4 - (\lambda_{41} + 2)(\lambda_{41} + 4)]x^2y - [\lambda_{21}(2 + \lambda_{41}) + (4 + \lambda_{41})\,(7\lambda_{21} - 4 - \lambda_{41})/4]\, xy^2 - \lambda_{21}\,(7\lambda_{21} - 4 - \lambda_{41})/4\, y^3\}\, dx + \lambda_{41}x\, \{-2\,(2 + \lambda_{41})x^3 - [2\lambda_{21}(2 + \lambda_{41}) + (7\lambda_{21} - 4 - \lambda_{41})/2]x^2y + [2 + \lambda_{41} - \lambda_{21}(7\lambda_{21} - 4 - \lambda_{41})/2]xy^2 + (7\lambda_{21} - 4 - \lambda_{41})y^3/4\}dy$. Then using the formulas in Problem 1(b), we find $q_3(x, y) = \lambda_{53}x - A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 - A_{03}y^3$ with $A_{30} = \lambda_{41}(2 + \lambda_{41})\,(7\lambda_{21} + 4 - \lambda_{41})/6$, $A_{21} = \lambda_{41}[\lambda_{21}(7\lambda_{21} - 4 - \lambda_{41})/4 - (\lambda_{41} + 2)\,(\lambda_{41} - 4)]$, $A_{12} = \lambda_{41}\,(7\lambda_{21} - 4 - \lambda_{41})/4$ and $A_{03} = \lambda_{41}[2(\lambda_{41} + 2)\,(\lambda_{41} + 5)/3 - \lambda_{21}(7\lambda_{21} - 4 - \lambda_{41})/4]$. And this allows us to compute $\alpha d_4(\alpha, \lambda) = M_4(\alpha) = \int_{H-h=\alpha^2/2} \Omega_4$ where $\Omega_4 = \omega_4 + q_1\omega_3 + q_2\omega_2 + q_3\omega_1$, $\omega_j = g_j\, dx - f_j\, dy$ (with $\omega_2 = 0$) and the $q_j(x, y)$, $j = 1, 2, 3$, are given above. This leads to the formula for $d_4(\alpha, \lambda)$ given in Lemma 1 in Section 4.11, using the fact that $\int_0^{2\pi} \sin^4 t \cos^2 t\, dt = \int_0^{2\pi} \sin^2 t \cos^4 t\, dt = \pi / 8$, $\int_0^{2\pi} \sin^6 t\, dt = \int_0^{2\pi} \cos^6 t\, dt = 5\pi / 8$ and using the integrals in parts (a) and (b).

In order to compute $d_5(h)$ and $d_6(h)$, it is necessary to obtain the formulas for Ω_4, q_4, Ω_5 and q_5 contained in [58]. That will not be done here.

PROBLEM SET 4.13

5. For the system (6) with the minus sign, $D\mathbf{f}(\mathbf{x}) = [0, 1; \mu_1 - 3x^2 - 2xy, \mu_2 - x^2]$. For $\mu_1 < 0$ the origin is the only critical point and $D\mathbf{f}(\mathbf{0}) = [0, 1; \mu_1, \mu_2]$; thus for $\mu_1 < 0$, the origin is a sink for $\mu_2 < 0$ and a source for $\mu_2 > 0$. Using equation (3′) in Section 4.4, we find $\sigma = -3\pi / 2|\mu_1|^{1/2} < 0$ and the origin is also stable for $\mu_2 = 0$. Furthermore, for $\mu_2 = 0$ there is a supercritical Hopf bifurcation at the origin in which a unique stable, negatively oriented limit cycle bifurcates from the origin as μ_2 increases from zero. It follows from the

rotated vector field theory in Section 4.6 that this stable limit cycle expands to infinity as μ_2 increases without bound. For $\mu_1 = 0$ the flow on the center manifold, $y = x^3/\mu_1 + 0(x^4)$, is given by $\dot{x} = x^3/\mu_1 + 0(x^4)$ and there is an unstable node at the origin. For $\mu_1 > 0$, we have critical points at the origin and at $(\pm\sqrt{\mu_1}, 0)$. Using equation (4) in Section 4.2, we can show that there is a pitchfork bifurcation for $\mu_1 = 0$ (and $\mu_2 \neq 0$) in which three critical points bifurcate from the origin as μ_1 increases through $\mu_1 = 0$. Since $\det Df(0) = -\mu_1 < 0$ for $\mu_1 > 0$, the origin is a saddle for $\mu_1 > 0$; and since $Df(\pm\sqrt{\mu_1}, 0) = [0,1;-2\mu_1, \mu_2 - \mu_1]$, $(\pm\sqrt{\mu_1}, 0)$ are sinks for $\mu_2 < \mu_1$ and sources for $\mu_2 > \mu_1$. After translating the origin to $(\pm\sqrt{\mu_1}, 0)$ and using equation (3′) in Section 4.4, we find that $\sigma = 3\pi/\sqrt{2\mu_1} > 0$ and thus, according to Theorem 1 in Section 4.4, there are subcritical Hopf bifurcations at $(\pm\sqrt{\mu_1}, 0)$ for $\mu_2 = \mu_1$ in which unique, negatively oriented, unstable limit cycles bifurcate from $(\pm\sqrt{\mu_1}, 0)$ as μ_2 decreases from μ_1. Using the rotated vector field theory in Section 4.6, it follows that these negatively oriented, unstable limit cycles expand as μ_2 decreases and (since this system is symmetric with respect to the origin) they simultaneously intersect the saddle at the origin and form a compound separatrix cycle with two loops at $\mu_2 = h(\mu_1)$. By making the rescaling transformation (7) we obtain the system (8). The system (8) was studied in Example 2 in Section 4.10. It follows from Theorem 5 in Section 4.10 that the homoclinic-loop bifurcation occurs at $\mu_2 = h(\mu_1) = 4\mu_1/5 + 0(\mu_1^2)$ as $\mu_1 \to 0^+$. Theorem 5 in Section 4.10 also establishes that for $\mu_1 > 0$ and $h(\mu_1) < \mu_2 < \mu_1$ there is exactly one limit cycle for this system around each of the critical points $(\pm\sqrt{\mu_1}, 0)$, neither of which encloses the origin. Finally, the computation of the Melnikov function for the exterior Duffing problem in Problem 6 of Section 4.10, which is contained in the original Solutions Manual, shows that there is a multiplicity-two limit cycle bifurcation surface given by $\mu_2 = c(\mu_1) \cong .752\mu_1 + 0(\mu_1^2)$ as $\mu_1 \to 0^+$. The bifurcation set and the corresponding phase portraits are shown below; cf. Figures 7.3.7 and 7.3.9 in [G/H].



This completes the appendix to the Solutions Manual for TAM 7, Differential Equations and Dynamical Systems. There is nothing to add concerning the Research Problems at the end of Section 4.15 since, to my knowledge, no further progress has been made on those problems beyond what is contained in Section 4.15 and in our Journal of Differential Equations paper [60], published in 2000.

5. SOME ADDITIONAL PROBLEMS

Problem 6 in Section 2.8: (Sternberg) Rewrite the system

$$\dot{x} = 2x + y^2$$

$$\dot{y} = y$$

in differential form as $dx/dy = 2x/y + y$ and solve this first-order, linear differential equation to obtain $x = y^2(c + \ln|y|)$. Solve the linearization of this system to obtain $x = c\,y^2$ and note that by Hartman's Theorem, the linear and nonlinear systems are C^1-diffeomorphic (or C^1- conjugate as defined in Section 3.1); in fact, all trajectories (except those on the x-axis) are tangent to the y-axis at the origin. However, the linear and nonlinear systems are not C^2 diffeomorphic since under a C^2 change of coordinates, the C^2 curves $x = cy^2$ would go into C^2 curves and the curves $x = y^2(c + \ln|y|)$ are not C^2. Note that the "resonance condition" $\lambda_1 = m_1\lambda_1 + m_2\lambda_2$ is satisfied with $m_1 = 0$, $m_2 = 2 > 1$, $\lambda_1 = 2$ and $\lambda_2 = 1$. Thus, neither Poincare's nor Sternberg's theorems apply.

Problem 8 in Section 2.12:
(a) Use Theorem 2 in Section 2.12 to find the approximation for the flow on the local center manifold, Γ, for the system

$$\dot{x} = xy + xz - x^4$$

$$\dot{y} = -y - x^2$$

$$\dot{z} = z + x^2.$$

And then sketch the local phase portrait for this system to see that the origin is a type of topological saddle in R^3 which is topologically equivalent to the saddle shown in Figure 3 of Section 1.1 (with $t \to -t$).
Hint: It can be shown that there is a smooth surface S: $z = \psi(x,y)$, containing the curve Γ and the y-axis such that for all $\underline{x}_0 \in S$, $\underline{\phi}_t(\underline{x}_0) \to 0$ as $t \to \infty$. See the figure below on the left.

(b) Use Theorem 2 in Section 2.12 to find the approximation for the flow on the local center manifold, Γ, for the system

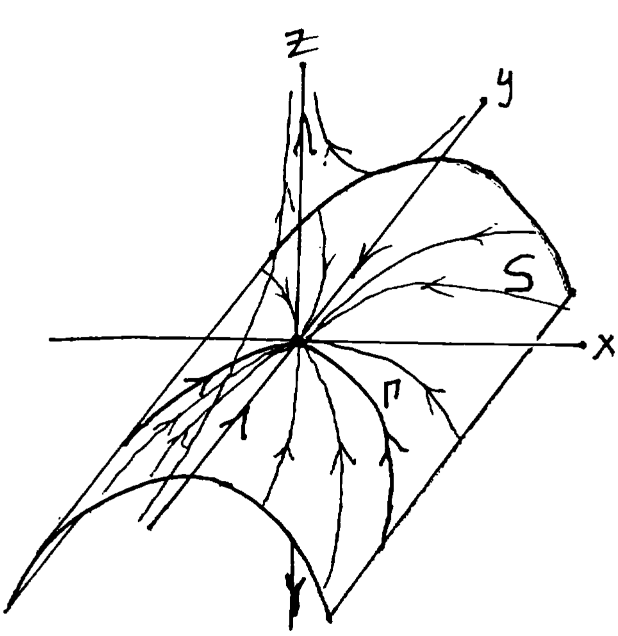
$$\dot{x} = x^2 y + x^2 z - x^5$$

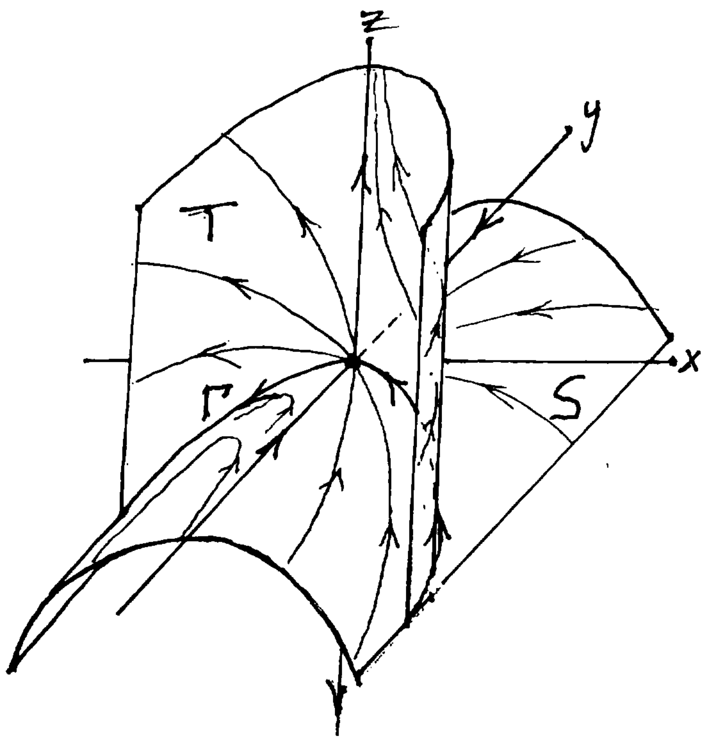
$$\dot{y} = -y - x^2$$

$$\dot{z} = z + x^2.$$

And then sketch the local phase portrait for this system to see that the origin is a type of saddle-node in R^3 for this system.

Hint: For $x \geqslant 0$ there is a surface S containing Γ and the y-axis as in part (a); for $x \leqslant 0$ there is a topological saddle on S at 0; i.e., there is a topological saddle-node on the surface S at the origin. There is another smooth surface T: $y = \gamma(x,z)$ containing the curve Γ and the z-axis such that for all $\underline{x}_0 = (x_0, y_0, z_0)$ with $x_0 \leqslant 0$, $\phi_t(\underline{x}_0) \to 0$ as $t \to -\infty$, and there is a topological saddle on T at 0 for $x_0 \geqslant 0$. See the figure below on the right.
Note: In part (a) we also have a surface T: $y = \gamma(x,z)$ containing Γ and the z-axis on which there is a topological saddle at 0.





Problem 8 in Section 4.2: (Khellat) Show that the system

$$\dot{x} = \mu x - xy$$
$$\dot{y} = -x + x^2$$

does not satisfy conditions (2), (3), or (4) in Section 4.2, but that it has a pitchfork bifurcation at the origin as the parameter μ varies through the bifurcation value $\mu = 0$. Thus, while condition (4) is sufficient for a pitchfork bifurcation, it is not necessary.

Problem 5 in Section 4.9: Compute the Melnikov function, $M(t_0)$, for the following perturbations of the undamped pendulum in Example 1 of Section 2.14 with Hamiltonian

$$H(x,y) = y^2/2 + 1 - \cos x$$

And show that $M(t_0)$ has simple zeros.

Note: Theorem 1 in Section 4.9 applies to the flow on the cylinder (obtained by identifying x-points mod 2π) in the following problems where, for $H = 2$, we have two homoclinic orbits $\Gamma_0^{\pm}$: $\gamma_0^{\pm}(t) = (x_0^{\pm}(t), y_0^{\pm}(t))$ at the hyperbolic saddle point $(\pi, 0) = (-\pi, 0)$ mod 2π. Thus, Lemma 1 implies that the Poincare map, P_ε, for the perturbed system has a unique hyperbolic fixed point $\mathbf{x}_\varepsilon = (\pi, 0) + 0(\varepsilon)$ of saddle type and Theorem 1 implies that the stable and unstable manifolds $W^s(\mathbf{x}_\varepsilon)$ and $W^u(\mathbf{x}_\varepsilon)$ of the Poincaré map intersect transversally. Therefore, we have the type of chaotic dynamics predicted by the Smale-Birkhoff Homoclinic Theorem. Cf. Figure 11 in Section 4.8 and Figure 9 on p. 158 in [15] which illustrate why an iterate P_ε^N of P_ε has a horseshoe map. Also, see p. 158 in [15] for an interesting discussion of why a periodically perturbed pendulum exhibits sensitive dependence on initial conditions.

(a)
$$\dot{x} = y$$
$$\dot{y} = -\sin x + \varepsilon \cos t$$

Hint: On the homoclinic orbits $\Gamma_0^{\pm}$ with $H = 2$, we have $\cos x = y^2/2 - 1$ and this allows us to integrate $\dot{y} = -\sin x = \pm\sqrt{1 - \cos^2 x} = \pm y\sqrt{1 - y^2/4}$ to obtain $y_0^{\pm}(t) = \pm 2 \operatorname{sech} t$. Then use the result of Problem 4 in Section 4.9 to find $M(t_0) = 2\pi \operatorname{sech}(\pi/2) \cos t_0$.

(b) (Poincare 1890; cf. [15], p.155)

$$\dot{x} = y$$

$$\dot{y} = -\sin x + \varepsilon \cos x \cos t.$$

Hint: Follow the hint for part (a), use two integrations by parts to evaluate

$$\int_{-\infty}^{\infty} \operatorname{sech} t \tanh^2 t \cos t \, dt$$

and then use the result of Problem 4 in Section 4.9 to find

$$M(t_0) = -2\pi \operatorname{sech}(\pi/2) \cos t_0$$

as on p. 157 in [15].

(c)

$$\dot{x} = y$$

$$\dot{y} = -\sin x + \varepsilon (\mu \cos t - y).$$

Hint: Similar to the result in Example 1 in Section 4.9, you should find that for $\mu > (4/\pi) \cosh(\pi/2)$, $M(t_0)$ has a simple zero.

Problem 7 in Section 4.15: Use Theorem 4 in Section 4.15 to show that for $\gamma = 0$ and $\beta < 0$, there is a point

$$TB_3^c : \alpha = \beta/2 - 1/\beta, \quad c = \beta/2$$

On the line TB_2^0 : $c = \alpha + 1/\beta$ (given in Theorem 3 in Section 4.9) which lies in the region $E = \{(\alpha, c) \mid \alpha > \beta, |c| < 2\}$ provided that $-4 < \beta < 0$. Cf. Figure 6 in Section 4.14 and note that the point TB_3^c has $\alpha > \beta$ and $-2 < c < 0$ for $-4 < \beta < 0$.

Note: For $-4 < \beta < 0$ and sufficiently small $\gamma > 0$, the unfolding of the TB_3^o point shown in the bifurcation diagram in Figure 1 in Section 4.15 implies that the points H_2^o, C_2^c and HL_2^o all lie in the region E in a small neighborhood of the TB_3^o point for small $\gamma > 0$.

Re-draw the charts in Figure 6 in Section 4.14 for $-4 < \beta < 0$ and small $\gamma > 0$, noting that the TB_2^c line bifurcates into the H^o line where $c = \alpha + (1 + \gamma^2)/\beta$ (given in Theorem 5 in Section 4.14) and the HL^c curve and that H^o and HL^o cross for $-4 < \beta < 0$ and $0 < \gamma << 1$ as in Figure 1 in Section 4.15. Also, note that the point H_2^o is on the H^o line, the point HL_2^c is on the HL^c curve and the C_2^o curve joins these two points as in Figure 1. Also, with the exception of the bifurcation of the TB_2^o line, described above, the other parts of the charts in Figure 6 in Section 4.14 remain the same.

One last comment: For $\beta \leq -4$ and $0 < \gamma << 1$, we have the point HL_2^o on the curve HL^c moving out of the region E and then the point H_2^o on the curve H^c moving out of the region E as β decreases from -4 (for points below the Γ_{14} curve shown in Figure A in the Appendix). This leads to Chart 1 in the Appendix.

Problem 13 in Section 2.[illegible]1 (Solving the two-body problem):

(a) The planar two-body problem can be written in the form

$$\ddot{\underline{r}} = -\underline{r}/r^3$$

where $\underline{r} = (x,y) = r(\cos\theta, \sin\theta)$ in polar coordinates. Show that

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{u}_\theta$$

where $\underline{u}_r = (\cos\theta, \sin\theta)$ and $\underline{u}_\theta = (-\sin\theta, \cos\theta)$ and that this implies that

$$\frac{d}{dt}(r^2\dot{\theta}) = 0,$$

i.e., that the angular momentum $r^2\dot{\theta} = h$, a constant. Then show that

$$\ddot{r} = h^2/r^3 - 1/r^2$$

This second order, nonlinear differential equation can be reduced to a second order, linear differential equation, that can be solved, by changing the dependent variable to $u = 1/r$ and by changing the independent variable to θ with $d/dt = hu^2 d/d\theta$. Show that this leads to

$$\frac{d^2u}{d\theta^2} + u = 1/h^2$$

and that the solution of this second order, linear differential equation can be written as $u(\theta) = (1 + e\cos\theta)/h^2$ where $e = -1 + h^2 r_p$. Thus, where $r_p = r(0) = 1/u(0)$.

$$r(\theta) = \frac{h^2}{(1 + e\cos\theta)},$$

a conic section with eccentricity e and periapsis distance r_p. The dependence of θ and r on t can be obtained from the inverse of the monotone function

$$t - t_0 = h^3 \int_0^\theta \frac{d\theta}{(1 + e\cos\theta)^2}$$

For $h \neq 0$.

(b) For $h = 0$ in the two-body problem, we have a linear motion described by

$$\ddot{r} = -1/r^2$$

or

$$\dot{v} = \frac{dv}{dr}v = -1/r^2$$

where $v = dr/dt$. Let $u = 1/r$ and show that $dv/dr = -u^2 dv/du$, i.e., that

$$v\, dv/du = 1.$$

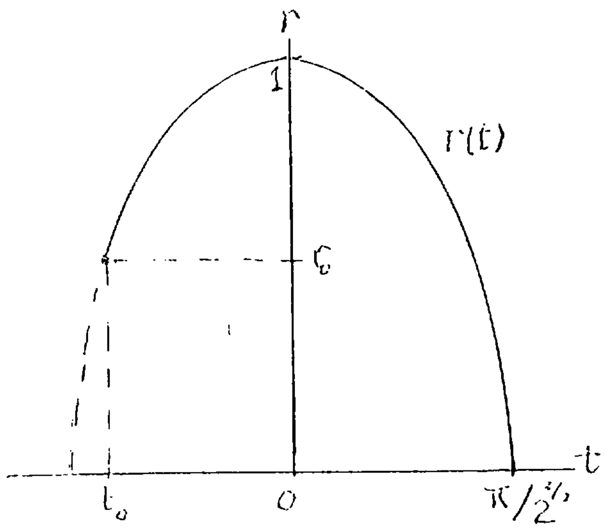
Thus, $v = \pm\sqrt{2(u - C_0)}$ where $C_0 = v_0^2/2 - 1/r_0$ is the total energy and this implies that

$$du/dt = \pm u^2\sqrt{2(u - |C_0|)}$$

for $C_0 < 0$. Integration then leads to

$$\pm\sqrt{2}\, t = \sqrt{r(1 - |C_0| r)}/|C_0| + 1/|C_0|^{3/2} \tan^{-1}\sqrt{(1 - |C_0| r)/|C_0| r}.$$

Graph these functions, $\pm t(r)$, and from these graphs, obtain the graph of the inverse function, $r(t)$. Note that from the graph of $r(t)$, it is clear that $t_0 < 0$ corresponds to $v_0 > 0$, $t_0 = 0$ corresponds to $v_0 = 0$, and $t_0 > 0$ corresponds to $v_0 < 0$ where for a given $r_0 > 0$ and $C_0 < 0$, $t_0 = \pm 1/\sqrt{2}\, [\sqrt{r_0(1 - |C_0| r_0)}/|C_0| + 1/|C_0|^{3/2} \tan^{-1}\sqrt{(1 - |C_0| r_0)/|C_0| r_0}\,]$ and $v_0 = \mp\sqrt{2(1/r_0 - |C_0|)}$.
For example, for $C_0 = -1$ and $t_0 < 0$, you should find $r(t)$ as shown below:



Similarly, for $C_0 > 0$ and $r_0 > 0$, you should find that

$$\pm\sqrt{2}\, t = \sqrt{r(1 + C_0 r)}/C_0 - 1/2C_0^{3/2} \ln\left[\frac{\sqrt{1 + C_0 r} - \sqrt{C_0 r}}{\sqrt{1 + C_0 r} + \sqrt{C_0 r}}\right].$$

Graph these functions, $\pm t(r)$, and from these graphs, obtain the graph of the inverse function $r(t)$. From the graph of $r(t)$, it is clear that $t_0 < 0$ corresponds to $v_0 < 0$ and $t_0 > 0$ corresponds to $v_0 > 0$ where $t_0 = \pm t(r_0)$ and $v_0 = \pm\sqrt{2(1/r_0 + C_0)}$.
Finally, for $C_0 = 0$, you should find $r(t) = 2^{1/3} t^{2/3}$ and that for a given $r_0 > 0$, $t_0 = \pm r_0^{3/2}/2^{1/2}$ and $v_0 = \pm\sqrt{2/r_0}$.

Problem 7′ in the Appendix:

(a) In order to see how the charts in Figure C change as γ increases for a fixed $\beta <<-1$, say $\beta= -10$, first of all note that for $\beta= -10$ and $\gamma= 3$, we obtain Chart 5 as in Figure 23 in Section 4.14 and for $\gamma= 3.5$, we obtain Chart 6 as in Figure 20 in Section 4.14; and then for $\gamma= 5.09$ graph the functions

$$H^+: \quad c = \frac{1+\alpha(\alpha+\beta+S)/2}{\alpha - S}$$

$$H^0: \quad c = \alpha + (1+\gamma^2)/\beta$$

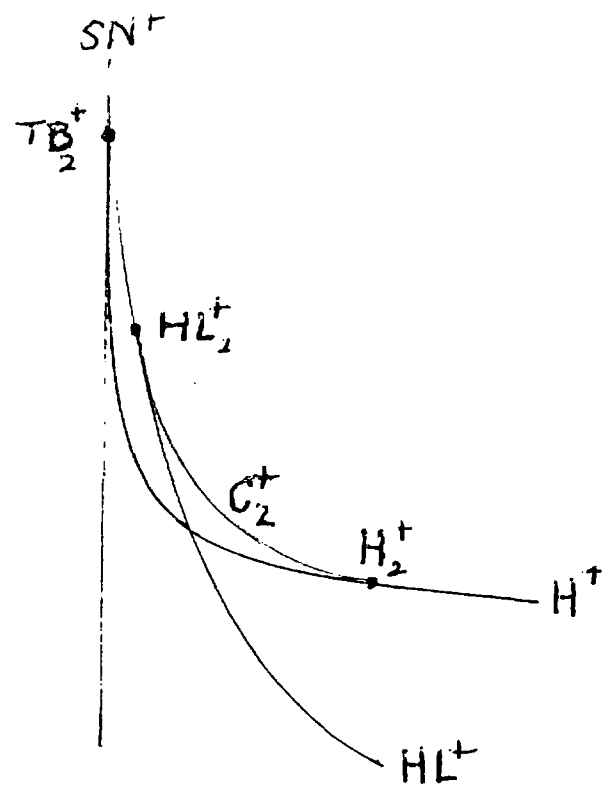
and $$SS: \quad c = (\alpha+\beta+S)/2$$

with $S=\sqrt{(\alpha-\beta)^2 - 4\gamma^2}$ obtained from Theorem 5 in Section 4.14. According to the atlas in Figure Q, for $\beta= -10$ and $\gamma= 5.09$, you should obtain Chart 7 since Γ_4^+, defined above, corresponds to $\gamma \cong 5.07$ and Γ_7^+, also defined above, corresponds to $\gamma = 5.1$. Then sketch in the HL^+ and C_2^+ curves to conform to Chart 7. Note that it was established in Section 4.13 that the H^+ and HL^+ curves approach the TB_2^+ point (given by

$$TB_2^+: \quad \alpha=\beta+2\gamma, \quad c = 1/(\beta+2\gamma)+\beta+\gamma$$

in Section 4.14) tangent to the SN^+ line ($\alpha=\beta+2\gamma$) with the H^+ curve between the HL^+ curve and the SN^+ line.

Hint: In order to see the detail near the TB_2^+ point, including the H_2^+ and HL_2^+ points as well as the C_2^+ curve (similar to that shown in Figure 20 in Section 4.14), it will be necessary to do an enlargement of the graphs near the TB_2^+ point similar to that shown in the figure below.



Next, graph the above functions H^+, H°, and SS for $\gamma = 5.11$. According to the atlas in Figure Q, for $\beta = -10$ and $\gamma = 5.11$, you should obtain Chart 8 since Γ_4^-, defined above, corresponds to $\gamma \cong 5.18$. Then sketch in the HL^+ and C_2^+ curves to conform to Chart 8.

Finally, graph the above functions H^+, H°, and SS for $\gamma = 5.1$ corresponding to a point on the TB_3^+ curve, Γ_7; and then sketch in the HL^+ and C_2^+ curves (similar to those in Chart 8), noting that, as in the last figure in this appendix, the H^+ and HL^+ curves are transverse to the SN^+ line at the point TB_3^+ as is established below where it is shown that for $\beta = -10$, $dc/d\alpha = -24.5$ at the point TB_3^+ on the line SN^+.

(b) From Theorem 4′ in Section 4.15, at the point TB_3^+ we have $\alpha = \beta + 2\gamma$ with $\gamma \neq 0$ (i.e., $TB_3^+ \in SN^+$), $(\beta + 2\gamma)(c - \alpha + \gamma) = 1$ (i.e., $TB_3^+ \subset TB_2^+$), and $\beta^2 + 2\beta\gamma + 2 = 0$ (i.e., the point $(\beta, \gamma) \in \Gamma_7$). Thus, at TB_3^+, $\gamma = (\beta^2 + 2)/2|\beta|$ and $\alpha = 2/|\beta|$ for $\beta < 0$. Show that on the curve

$\hat{c} = 1/|\beta|$

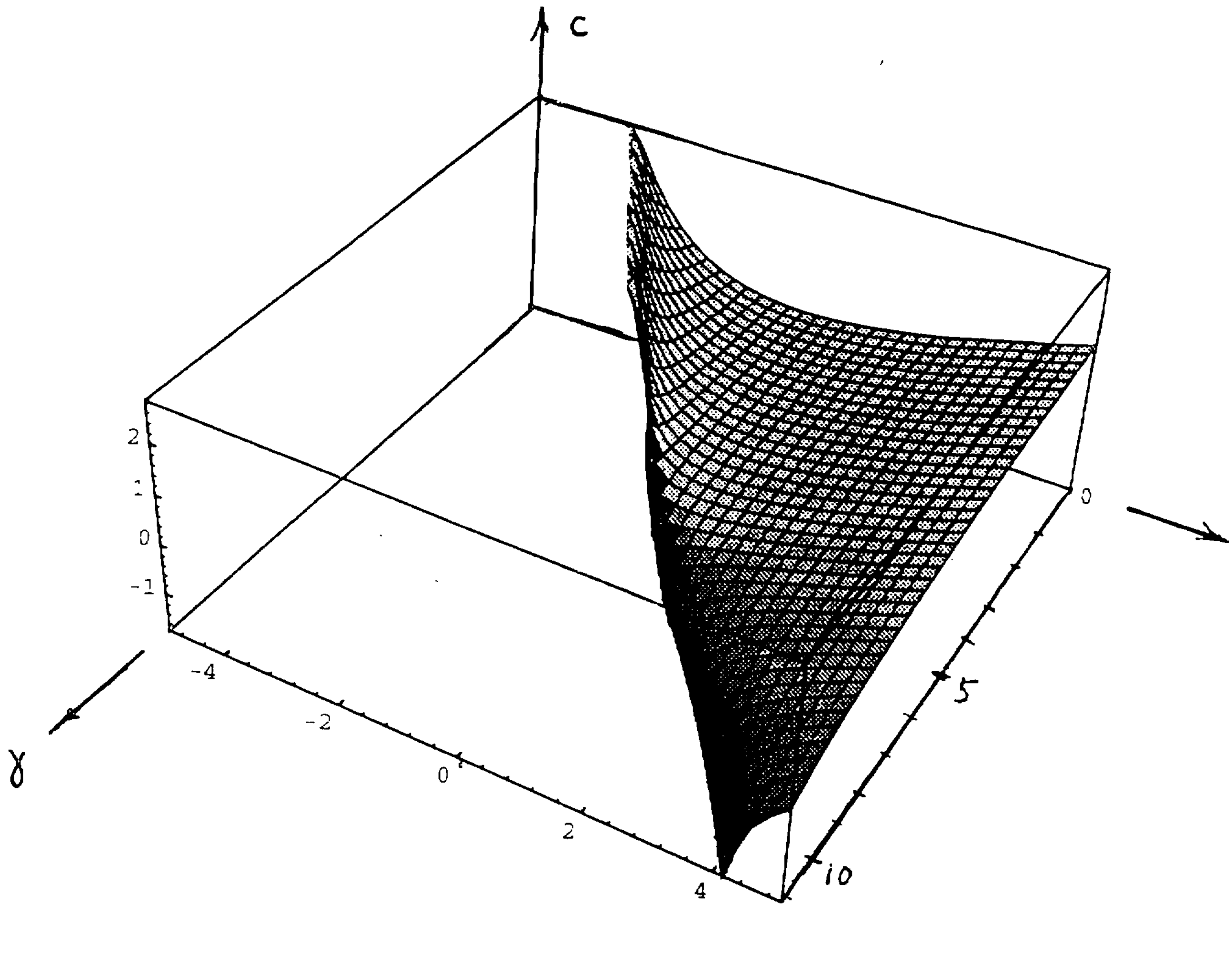
$$H^+: \quad c = \frac{1 + \alpha(\alpha - \beta + S)/2}{\alpha + S}$$

with $S = \sqrt{(\alpha - \beta)^2 - 4\gamma^2}$, we have

$$\frac{dc}{d\alpha} \to \frac{2 - \beta^2}{4}$$

as $\alpha \to 2/|\beta|$ for a fixed $\beta < 0$ and $\gamma = (\beta^2 + 2)/2|\beta|$.

Hint: Note that $S = 0$ at TB_3^+; so it is necessary to use L'Hospital's Rule for that part of $dc/d\alpha$ having the form $0/0$ for a fixed $\beta < 0$ and $\gamma = (\beta^2 + 2)/2|\beta|$ as $\alpha \to 2/|\beta|$.



The Hopf Bifurcation Surface H^+ for $\beta = -10$ with $\gamma > 0$ and $|c| < 2$

6. ADDITIONS AND CORRECTIONS

p. 217, l. 27: $\cdots e^{Bt}$

p. 239, l. 39: has at most a finite number of critical points in any compact subset

p. 261, l. 33: of a finite number of elementary critical points (i.e., critical points with at least one nonzero eigenvalue) on the equator

p. 272, l. 12, 13: And for $z = 0$ we have $-\dot{x} = x - (\alpha + 1)x^2 + x^3$ in (10). Therefore, $\dot{x} > 0$ for $x < 0$ and $\dot{x} < 0$ for $x > 0$ on the x-axis. $\cdots$

p. 279, l. 23: around the unit circle with velocity $\underline{v} = \dot{\underline{\phi}}_t(\underline{p})$ tangent to C at the point $\underline{\phi}_t(\underline{p})$.

p. 285, l. 7: having a nonzero eigenvalue (to which the theorems in Section 2.8 ....

p. 408, l. 23: (2) has a unique hyperbolic periodic orbit of saddle type, $\underline{x}_\varepsilon(t) = \underline{x}_0 + O(\varepsilon)$, of period T. [l. 20: Replace $\underline{\gamma}_\varepsilon(t)$ by $\underline{x}_\varepsilon(t)$.]

p. 409: Replace $\underline{\gamma}_\varepsilon(t)$ by $\underline{x}_\varepsilon(t)$ in Figure2.

p. 417, l. 14,15:

$$\pm\dot{x} = x^2 z^2 - x^4 + \varepsilon\alpha x z^2 + \varepsilon\beta x^3 - z^2$$
$$\pm\dot{z} = x z^3 - x^3 z + \varepsilon\alpha z^3 + \varepsilon\beta x^2 z.$$

p. 417, l. 18, 19: Replace $(\beta,0)$ by $(\varepsilon\beta,0)$.

p. 428, l. 10: Replace n-dimensional by m-dimensional

p. 433, l. 4: Replace $\underline{f}(\underline{\gamma}_0(t))$ by $\underline{f}(\underline{\gamma}_\alpha(t))$.

p. 443, l. 33, 34: Replace (4) by (1).

p. 445, l. 6: Replace $P_2(x,y,\mu)$ by $P_2(x,h,\mu)$.

p. 480, l. 17: accomplished for $\beta \geq 0$ in Figures 15 and 16 below and in the Appendix at the end of Section 4.15 for $\beta << -1$ (e.g., $\beta < -5$); but for $-5 < \beta < 0$ it is still ...

p. 346, l. 13: $\cdots + a_{11} b_{20})$

p. 484: l. 12: ... to (3) with $a_{11} = \dots$

p. 486: Replace .1 by -1 in Figure 4.

p. 493, l. 34: below for $\beta \geq 0$ and in Figures A and C (for $\beta << -1$) in the Appendix at the end of Section 4.15 ....

p. 495, l. 16: .... for $\beta \geq 0$ and in the Appendix at the end of Section 4.15 for $\beta << -1$.
l. 23: The "atlas" shown in Figure A in [53] and in the Appendix at the end of Section 4.15 (as well in Figure 15 below for $\beta \geq 0$).

p. 502, l.14: together with the atlas and charts in [53] and in the Appendix can be used ...
l.21: values of β, as described in the Appendix, and also ....
l.26: 19, 20, and 22 below and also in Chart 6 in the Appendix at the end of Section 4.15 ....

p. 503, l. 11: scale in Figure 20 which is the same as Chart 6 in the Appendix, ...

p. 505, l. 8: .... it has a flat contact with HL^c at HL^c_2 (which is outside the region E in this example). Cf. Chart 6 in the Appendix.
l.11: .... 23. Cf. Chart 5 in the Appendix. We see ....

p. 506, l.14: .... affinely equivalent to the BQS with

p. 512, l.10: are shown in Figures 15 and 16 for $\beta \geq 0$ and in the Appendix at the end of Section 4.15 for $\beta << -1$.

p. 528, l. 6: $\dot{\underline{x}} = \underline{f}(\underline{x}, \mu)$

p. 539, l. 16,17: .... Note that in the region between HL^+ and C^+_2 in Figure C, ...
l. 37: ..... it terminates at N_2. This leads to

p. 540, l. 1: .... It also terminates at N_2.